Problem 8

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

$$\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$$

Solution

Apply the ratio test to determine the condition in which the provided series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}}}{\frac{n!x^n}{n^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}}\frac{x^{n+1}}{x^n}\right| \\
&= \lim_{n\to\infty}\left|(n+1)\frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}x\right| \\
&= \lim_{n\to\infty}\left|(n+1)\frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}x\right| \\
&= \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}|x| \\
&= \left(\lim_{n\to\infty}\frac{n+1}{n}\right)\left[\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}\right]|x| \\
&= \left(\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1}\right)\left(\frac{1}{e}\right)|x| \\
&= (1)\left(\frac{1}{e}\right)|x| \\
&= \frac{1}{e}|x|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } \frac{1}{e}|x| < 1 \\ \text{unknown} & \text{if } \frac{1}{e}|x| = 1 \\ \text{divergent} & \text{if } \frac{1}{e}|x| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x| < e$, or $-e < x < e$, we see that the center of convergence is at $x = 0$ and the radius of convergence is e.