Problem 9

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\sin x, \qquad x_0 = 0$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 0$, this formula reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The aim then is to determine the nth derivative of $\sin x$ evaluated at $x = 0$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0: &\quad f^{(0)}(x) = \sin x &\quad\to\quad& f^{(0)}(0) = 0 \\
n = 1: &\quad f^{(1)}(x) = \cos x &\quad\to\quad& f^{(1)}(0) = 1 \\
n = 2: &\quad f^{(2)}(x) = -\sin x &\quad\to\quad& f^{(2)}(0) = 0 \\
n = 3: &\quad f^{(3)}(x) = -\cos x &\quad\to\quad& f^{(3)}(0) = -1 \\
n = 4: &\quad f^{(4)}(x) = \sin x &\quad\to\quad& f^{(4)}(0) = 0 \\
n = 5: &\quad f^{(5)}(x) = \cos x &\quad\to\quad& f^{(5)}(0) = 1 \\
&\quad \vdots
\end{aligned}$$

Since $f^{(0)}(0) = 0$, the sum can be started from $n = 1$.

$$f(x) = \sin x = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

All the even values of n result in $f^{(n)}(0)$ being zero, so sum over the odd integers only by making the substitution $n = 2k + 1$.

$$\begin{aligned}
\sin x &= \sum_{2k+1=1}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1}
\end{aligned}$$

Based on the results above, $f^{(2k+1)}(0) = (-1)^k$. Therefore,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}}{[2(k+1)+1]!}x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!}x^{2k+1}}\right| \\
&= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(-1)^k}\frac{(2k+1)!}{(2k+3)!}\frac{x^{2k+3}}{x^{2k+1}}\right| \\
&= \lim_{k\to\infty}\left|-\frac{1}{(2k+3)(2k+2)}x^2\right| \\
&= \lim_{k\to\infty}\frac{1}{(2k+3)(2k+2)}x^2 \\
&= 0x^2
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } 0x^2 < 1 \\ \text{unknown} & \text{if } 0x^2 = 1 \\ \text{divergent} & \text{if } 0x^2 > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x| < \sqrt{1/0} = \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius of convergence is ∞.