Problem 10

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$e^x, \qquad x_0 = 0$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 0$, this formula reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The aim then is to determine the nth derivative of e^x evaluated at $x = 0$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0:& \quad f^{(0)}(x) = e^x & \rightarrow & \quad f^{(0)}(0) = 1 \\
n = 1:& \quad f^{(1)}(x) = e^x & \rightarrow & \quad f^{(1)}(0) = 1 \\
n = 2:& \quad f^{(2)}(x) = e^x & \rightarrow & \quad f^{(2)}(0) = 1 \\
n = 3:& \quad f^{(3)}(x) = e^x & \rightarrow & \quad f^{(3)}(0) = 1 \\
n = 4:& \quad f^{(4)}(x) = e^x & \rightarrow & \quad f^{(4)}(0) = 1 \\
n = 5:& \quad f^{(5)}(x) = e^x & \rightarrow & \quad f^{(5)}(0) = 1 \\
& \quad \vdots & &
\end{aligned}$$

Notice that $f^{(n)}(0) = 1$ for all n. Therefore,

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \frac{x^{n+1}}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{n} x \right| \\
&= \lim_{n\to\infty} \frac{1}{n} |x| \\
&= 0|x|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } 0|x| < 1 \\ \text{unknown} & \text{if } 0|x| = 1 \\ \text{divergent} & \text{if } 0|x| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x| < 1/0 = \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius of convergence is ∞.