Problem 11

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$x, \qquad x_0 = 1$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 1$, this formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x - 1)^n.$$

The aim then is to determine the nth derivative of x evaluated at $x = 1$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0: &\quad f^{(0)}(x) = x && \rightarrow && f^{(0)}(1) = 1 \\
n = 1: &\quad f^{(1)}(x) = 1 && \rightarrow && f^{(1)}(1) = 1 \\
n = 2: &\quad f^{(2)}(x) = 0 && \rightarrow && f^{(2)}(1) = 0 \\
n = 3: &\quad f^{(3)}(x) = 0 && \rightarrow && f^{(3)}(1) = 0 \\
&\quad \vdots
\end{aligned}$$

Notice that $f^{(n)}(1) = 0$ for $n > 1$. Therefore,

$$\begin{aligned}
f(x) = x &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\
&= (x-1)^0 + (x-1)^1 + \sum_{n=2}^{\infty} \frac{0}{n!}(x-1)^n \\
&= 1 + (x-1).
\end{aligned}$$

Since this series is finite, it is convergent for any value of x. That is, the radius of convergence is ∞.