Problem 15

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\frac{1}{1-x}, \qquad x_0 = 0$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 0$, this formula reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The aim then is to determine the nth derivative of $1/(1-x)$ evaluated at $x = 0$. Start taking derivatives and try to observe a pattern.

$$\begin{array}{llcl}
n = 0: & f^{(0)}(x) = (1-x)^{-1} & \to & f^{(0)}(0) = 1 \\
n = 1: & f^{(1)}(x) = (1-x)^{-2} & \to & f^{(1)}(0) = 1 \\
n = 2: & f^{(2)}(x) = 2(1-x)^{-3} & \to & f^{(2)}(0) = 2 \\
n = 3: & f^{(3)}(x) = 6(1-x)^{-4} & \to & f^{(3)}(0) = 6 \\
n = 4: & f^{(4)}(x) = 24(1-x)^{-5} & \to & f^{(4)}(0) = 24 \\
 & \vdots & &
\end{array}$$

Notice that $f^{(n)}(0) = n!$.

$$\begin{aligned}
f(x) = \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{n!}{n!}x^n \\
&= \sum_{n=0}^{\infty} x^n
\end{aligned}$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \\
&= \lim_{n\to\infty} |x| \\
&= |x|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |x| < 1 \\ \text{unknown} & \text{if } |x| = 1 \\ \text{divergent} & \text{if } |x| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < x < 1$, we see that the center of convergence is at $x = 0$ and the radius of convergence is 1.