Problem 18

Given that $y = \sum_{n=0}^{\infty} a_n x^n$, compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term. Show that if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1. Show that $a_{n+2} = a_n/(n+2)(n+1)$, $n = 0, 1, 2, 3, \ldots$.

Solution

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots + a_n x^n + \cdots \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots + n a_n x^{n-1} + \cdots \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots + n(n-1) a_n x^{n-2} + \cdots \end{aligned}$$

Suppose that $y'' = y$. Then

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute $n = k$ in the series on the left and substitute $n = k + 2$ in the series on the right.

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k+2=2}^{\infty} (k+2)(k+2-1) a_{k+2} x^{(k+2)-2}$$

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$

The coefficients must be equal.

$$a_k = (k+2)(k+1) a_{k+2}$$

Therefore, solving for a_{k+2},

$$\boxed{a_{k+2} = \frac{a_k}{(k+1)(k+2)}, \quad k = 0, 1, 2, \ldots.}$$

Starting with $k = 0$,

$$a_2 = \frac{a_0}{2},$$

and incrementing by 2 each time,

$$\begin{aligned} a_4 &= \frac{a_2}{12} = \frac{a_0}{24} \\ a_6 &= \frac{a_4}{30} = \frac{a_0}{720}, \\ &\vdots \end{aligned}$$

we see that a_k is in terms of a_0, an arbitrary constant, if k is even. On the other hand, starting with $k = 1$,

$$a_3 = \frac{a_1}{6},$$

and incrementing by 2 each time,

$$\begin{aligned} a_5 &= \frac{a_3}{20} = \frac{a_1}{120} \\ a_7 &= \frac{a_5}{42} = \frac{a_1}{5040}, \\ &\vdots \end{aligned}$$

we see that a_k is in terms of a_1, another arbitrary constant, if k is odd.