Problem 13

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\ln x, \qquad x_0 = 1$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 1$, this formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x - 1)^n.$$

The aim then is to determine the nth derivative of $\ln x$ evaluated at $x = 1$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0: &\quad f^{(0)}(x) = \ln x & \rightarrow &\quad f^{(0)}(1) = 0 \\
n = 1: &\quad f^{(1)}(x) = x^{-1} & \rightarrow &\quad f^{(1)}(1) = 1 \\
n = 2: &\quad f^{(2)}(x) = -x^{-2} & \rightarrow &\quad f^{(2)}(1) = -1 \\
n = 3: &\quad f^{(3)}(x) = 2x^{-3} & \rightarrow &\quad f^{(3)}(1) = 2 \\
n = 4: &\quad f^{(4)}(x) = -6x^{-4} & \rightarrow &\quad f^{(4)}(1) = -6 \\
n = 5: &\quad f^{(5)}(x) = 24x^{-5} & \rightarrow &\quad f^{(5)}(1) = 24 \\
&\quad \vdots & &
\end{aligned}$$

Since $f^{(0)}(1) = 0$, the series can start at $n = 1$.

$$f(x) = \ln x = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x - 1)^n.$$

Notice that $f^{(n)}(1) = (-1)^{n-1}(n-1)!$ for $n > 0$.

$$\begin{aligned}
\ln x &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x - 1)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x - 1)^n
\end{aligned}$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^n}{n+1}(x-1)^{n+1}}{\frac{(-1)^{n-1}}{n}(x-1)^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)\frac{n}{n+1}(x-1)\right| \\
&= \lim_{n\to\infty}\frac{n}{n+1}|x-1| \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x-1| \\
&= |x-1|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |x-1|<1 \\ \text{unknown} & \text{if } |x-1|=1 \\ \text{divergent} & \text{if } |x-1|>1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < x - 1 < 1$, or $0 < x < 2$, we see that the center of convergence is at $x = 1$ and the radius of convergence is 1.