Problem 14

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\frac{1}{1+x}, \qquad x_0 = 0$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 0$, this formula reduces to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The aim then is to determine the nth derivative of $1/(1+x)$ evaluated at $x = 0$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0: &\quad f^{(0)}(x) = (1+x)^{-1} && \rightarrow && f^{(0)}(0) = 1 \\
n = 1: &\quad f^{(1)}(x) = -(1+x)^{-2} && \rightarrow && f^{(1)}(0) = -1 \\
n = 2: &\quad f^{(2)}(x) = 2(1+x)^{-3} && \rightarrow && f^{(2)}(0) = 2 \\
n = 3: &\quad f^{(3)}(x) = -6(1+x)^{-4} && \rightarrow && f^{(3)}(0) = -6 \\
n = 4: &\quad f^{(4)}(x) = 24(1+x)^{-5} && \rightarrow && f^{(4)}(0) = 24 \\
&\quad \vdots
\end{aligned}$$

Notice that $f^{(n)}(0) = (-1)^n n!$.

$$\begin{aligned}
f(x) = \frac{1}{1+x} &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}x^n \\
&= \sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(-1)^n x^n}\right| \\
&= \lim_{n\to\infty} |(-1)x| \\
&= \lim_{n\to\infty} |x| \\
&= |x|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |x| < 1 \\ \text{unknown} & \text{if } |x| = 1 \\ \text{divergent} & \text{if } |x| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < x < 1$, we see that the center of convergence is at $x = 0$ and the radius of convergence is 1.