Problem 17

Given that $y = \sum_{n=0}^{\infty} nx^n$, compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term.

Solution

$$y(x) = \sum_{n=0}^{\infty} nx^n = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots$$
$$y'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \cdots + n^2 x^{n-1} + \cdots$$
$$y''(x) = \sum_{n=2}^{\infty} n^2(n-1)x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \cdots + n^2(n-1)x^{n-2} + \cdots$$

Note that the two series for y' and y'' can be made to start from zero by making the substitutions, $n = k + 1$ and $n = m + 2$, respectively.

$$y'(x) = \sum_{k+1=1}^{\infty} (k+1)^2 x^{(k+1)-1} = \sum_{k=0}^{\infty} (k+1)^2 x^k$$
$$y''(x) = \sum_{m+2=2}^{\infty} (m+2)^2(m+2-1)x^{(m+2)-2} = \sum_{m=0}^{\infty} (m+2)^2(m+1)x^m$$