Problem 28

Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

Solution

Substitute $k = n - 1$ in the first sum and substitute $k = n$ in the second sum.

$$\sum_{k+1=1}^{\infty} (k+1)a_{k+1}x^k + 2\sum_{k=0}^{\infty} a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Combine the two sums.

$$\sum_{k=0}^{\infty}[(k+1)a_{k+1}x^k + 2a_k x^k] = 0$$

Factor x^k.

$$\sum_{k=0}^{\infty}[(k+1)a_{k+1} + 2a_k]x^k = 0$$

The coefficients must be zero.

$$(k+1)a_{k+1} + 2a_k = 0$$

Solve for a_{k+1}.

$$a_{k+1} = -\frac{2}{k+1}a_k$$

This is a first-order linear difference equation that can be solved by iteration.

$$\begin{aligned}
k = 0: &\quad a_1 = -\frac{2}{1}a_0 \\
k = 1: &\quad a_2 = -\frac{2}{2}a_1 = -\frac{2}{2}\left(-\frac{2}{1}\right)a_0 \\
k = 2: &\quad a_3 = -\frac{2}{3}a_2 = -\frac{2}{3}\left(-\frac{2}{2}\right)\left(-\frac{2}{1}\right)a_0 \\
k = 3: &\quad a_4 = -\frac{2}{4}a_3 = -\frac{2}{4}\left(-\frac{2}{3}\right)\left(-\frac{2}{2}\right)\left(-\frac{2}{1}\right)a_0 \\
&\quad \vdots \\
&\quad a_k = \frac{(-2)^k}{k!}a_0
\end{aligned}$$

The function represented by the series $\sum_{n=0}^{\infty} a_n x^n$ is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 e^{-2x}.$$