Problem 5

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$$

Solution

Apply the ratio test to determine the condition in which the provided series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(2x+1)^{n+1}}{(n+1)^2}}{\frac{(2x+1)^n}{n^2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2}(2x+1) \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2}{n^2\left(1+\frac{1}{n}\right)^2}(2x+1) \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{\left(1+\frac{1}{n}\right)^2}(2x+1) \right| \\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2}|2x+1| \\
&= |2x+1|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |2x+1| < 1 \\ \text{unknown} & \text{if } |2x+1| = 1 \\ \text{divergent} & \text{if } |2x+1| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < 2x+1 < 1$, or $-1 < x < 0$, we see that the center of convergence is at $x = -1/2$ and the radius of convergence is $1/2$.