Problem 1

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' - y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and substitute $k = n$ in the second sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - a_k] x^k = 0$$

The coefficients must then be zero.

$$(k+2)(k+1) a_{k+2} - a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{a_k}{(k+2)(k+1)}$$

Plug in different values of k to determine a pattern for a_k.

$$\begin{aligned} a_2 &= \frac{a_0}{2 \cdot 1} & a_3 &= \frac{a_1}{3 \cdot 2} \\ a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} & a_5 &= \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} \\ a_6 &= \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} & a_7 &= \frac{a_5}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \\ &\vdots & &\vdots \\ a_{2k} &= \frac{a_0}{(2k)!} & a_{2k+1} &= \frac{a_1}{(2k+1)!} \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n \text{ even}}^{\infty} a_n x^n + \sum_{n \text{ odd}}^{\infty} a_n x^n \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{a_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{a_1}{(2k+1)!} x^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \\ &= a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right) + a_1 \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right) \\ &= a_0 \cosh x + a_1 \sinh x. \end{aligned}$$

Calculate the Wronskian of $\cosh x$ and $\sinh x$ now.

$$W(\cosh x, \sinh x) = \begin{vmatrix} \cosh x & \sinh x \\ (\cosh x)' & (\sinh x)' \end{vmatrix} = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1$$

Since the Wronskian is not zero at $x = 0$, the two functions, $\cosh x$ and $\sinh x$, form a fundamental set of solutions for the ODE.