Problem 4

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' + k^2x^2y = 0, \qquad x_0 = 0, \quad k \text{ a constant}$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + k^2x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring k^2x^2 into the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

Substitute $p = n - 4$ in the first sum and $p = n$ in the second sum to make the factors of x the same.

$$\sum_{p+4=2}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Solve for p.

$$\sum_{p=-2}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Write out the first two terms of the first sum.

$$(2)(1)a_2 + (3)(2)a_3 x + \sum_{p=0}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Now that the limits of each sum are the same and they have the same factors of x, the sums can be combined.

$$2a_2 + 6a_3x + \sum_{p=0}^{\infty}[(p+4)(p+3)a_{p+4}x^{p+2} + k^2a_px^{p+2}] = 0$$

Factor the summand.

$$2a_2 + 6a_3x + \sum_{p=0}^{\infty}[(p+4)(p+3)a_{p+4} + k^2a_p]x^{p+2} = 0 + 0x + 0x^2 + \cdots$$

Match the coefficients on both sides of the equation.

$$\begin{aligned} 2a_2 &= 0 \\ 6a_3 &= 0 \\ (p+4)(p+3)a_{p+4} + k^2a_p &= 0 \end{aligned}$$

As a result,

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_{p+4} &= -\frac{k^2}{(p+4)(p+3)}a_p. \end{aligned}$$

Plug in enough values of p to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned} p = 2: &\quad a_6 = -\frac{k^2}{6\cdot 5}a_2 = 0 & p = 3: &\quad a_7 = -\frac{k^2}{7\cdot 6}a_3 = 0 \\ p = 6: &\quad a_{10} = -\frac{k^2}{10\cdot 9}a_6 = 0 & p = 7: &\quad a_{11} = -\frac{k^2}{11\cdot 10}a_7 = 0 \\ p = 10: &\quad a_{14} = -\frac{k^2}{14\cdot 13}a_{10} = 0 & p = 11: &\quad a_{15} = -\frac{k^2}{15\cdot 14}a_{11} = 0 \\ p = 14: &\quad a_{18} = -\frac{k^2}{18\cdot 17}a_{14} = 0 & p = 15: &\quad a_{19} = -\frac{k^2}{19\cdot 18}a_{15} = 0 \\ &\quad \vdots & &\quad \vdots \\ &\quad a_{4m+2} = 0 & &\quad a_{4m+3} = 0 \end{aligned}$$

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\begin{aligned}
p = 0: &\quad a_4 = -\frac{k^2}{4\cdot 3}a_0 \\
p = 4: &\quad a_8 = -\frac{k^2}{8\cdot 7}a_4 = \frac{k^4}{8\cdot 7\cdot 4\cdot 3}a_0 \\
p = 8: &\quad a_{12} = -\frac{k^2}{12\cdot 11}a_8 = -\frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}a_0 \\
p = 12: &\quad a_{16} = -\frac{k^2}{16\cdot 15}a_{12} = \frac{k^8}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}a_0 \\
&\quad \vdots \\
&\quad a_{4m} = (-1)^m \frac{k^{2m}}{[(4m)(4m-4)\cdots(4)][(4m-1)(4m-5)\cdots(3)]}a_0 \\
&\quad\quad = (-1)^m \frac{k^{2m}}{[4^m(m)(m-1)\cdots(1)]\left[4^m\left(m-\frac{1}{4}\right)\left(m-\frac{5}{4}\right)\cdots\left(\frac{3}{4}\right)\right]}a_0 \\
&\quad\quad = (-1)^m \frac{k^{2m}}{[4^m m!]\left[4^m\frac{\Gamma\left(m+\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]}a_0 \\
&\quad\quad = (-1)^m \frac{k^{2m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)}a_0 \\
&\quad\quad = \frac{(-1)^m}{m!}\left(\frac{k}{4}\right)^{2m}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)}a_0
\end{aligned}
$$

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\begin{aligned}
p = 1: &\quad a_5 = -\frac{k^2}{5\cdot 4}a_1 \\
p = 5: &\quad a_9 = -\frac{k^2}{9\cdot 8}a_5 = \frac{k^4}{9\cdot 8\cdot 5\cdot 4}a_1 \\
p = 9: &\quad a_{13} = -\frac{k^2}{13\cdot 12}a_9 = -\frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}a_1 \\
p = 13: &\quad a_{17} = -\frac{k^2}{17\cdot 16}a_{13} = \frac{k^8}{17\cdot 16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}a_1 \\
&\quad \vdots \\
&\quad a_{4m+1} = (-1)^m \frac{k^{2m}}{[(4m+1)(4m-3)\cdots(5)][(4m)(4m-4)\cdots(4)]}a_1 \\
&\quad\quad = (-1)^m \frac{k^{2m}}{\left[4^m\left(m+\frac{1}{4}\right)\left(m-\frac{3}{4}\right)\cdots\left(\frac{5}{4}\right)\right][4^m(m)(m-1)\cdots(1)]}a_1 \\
&\quad\quad = (-1)^m \frac{k^{2m}}{\left[4^m\frac{\Gamma\left(m+\frac{5}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\right][4^m m!]}a_1 \\
&\quad\quad = (-1)^m \frac{k^{2m}\Gamma\left(\frac{5}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{5}{4}\right)}a_1 \\
&\quad\quad = \frac{(-1)^m}{m!}\left(\frac{k}{4}\right)^{2m}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(m+\frac{5}{4}\right)}a_1
\end{aligned}
$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{m=0}^{\infty} a_{4m} x^{4m} + \sum_{m=0}^{\infty} a_{4m+1} x^{4m+1} + \sum_{m=0}^{\infty} a_{4m+2} x^{4m+2} + \sum_{m=0}^{\infty} a_{4m+3} x^{4m+3} \\
&= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)} x^{4m} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(m+\frac{5}{4}\right)} x^{4m+1} \\
&= a_0 \left(1 - \frac{k^2}{4\cdot 3} x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3} x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{12} + \frac{k^8}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{16} - \cdots\right) \\
&\quad + a_1 \left(x - \frac{k^2}{5\cdot 4} x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4} x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{13} + \frac{k^8}{17\cdot 16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{17} - \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(1 - \frac{k^2}{4\cdot 3} x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3} x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{12} + \frac{k^8}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{16} - \cdots\right) \\
&\qquad \times \left(1 - \frac{k^2}{4} x^4 + \frac{k^4}{8\cdot 5\cdot 4} x^8 - \frac{k^6}{12\cdot 9\cdot 8\cdot 5\cdot 4} x^{12} + \frac{k^8}{16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{16} - \cdots\right) \\
&\qquad - \left(-\frac{k^2}{3} x^3 + \frac{k^4}{7\cdot 4\cdot 3} x^7 - \frac{k^6}{11\cdot 8\cdot 7\cdot 4\cdot 3} x^{11} + \frac{k^8}{15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{15} - \cdots\right) \\
&\qquad\quad \times \left(x - \frac{k^2}{5\cdot 4} x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4} x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{13} + \frac{k^8}{17\cdot 16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{17} - \cdots\right)
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.