Problem 7

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' + xy' + 2y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and 2 into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the other sums.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Solve for k in the first sum. The second sum can be set to start from $k = 0$ because k is present in the summand.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k + k a_k x^k + 2 a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + ka_k + 2a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + (k+2)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}(k+2)[(k+1)a_{k+2} + a_k]x^k = 0$$

The coefficients must be zero.

$$(k+1)a_{k+2} + a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = -\frac{a_k}{k+1}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = -\frac{a_0}{1} \\
k=1: &\quad a_3 = -\frac{a_1}{2} \\
k=2: &\quad a_4 = -\frac{a_2}{3} = \frac{a_0}{3\cdot 1} \\
k=3: &\quad a_5 = -\frac{a_3}{4} = \frac{a_1}{4\cdot 2} \\
k=4: &\quad a_6 = -\frac{a_4}{5} = -\frac{a_0}{5\cdot 3\cdot 1} \\
k=5: &\quad a_7 = -\frac{a_5}{6} = -\frac{a_1}{6\cdot 4\cdot 2} \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x - \frac{a_0}{1}x^2 - \frac{a_1}{2}x^3 + \frac{a_0}{3\cdot 1}x^4 + \frac{a_1}{4\cdot 2}x^5 - \frac{a_0}{5\cdot 3\cdot 1}x^6 - \frac{a_1}{6\cdot 4\cdot 2}x^7 + \cdots \\
&= a_0\left(1 - \frac{x^2}{1} + \frac{x^4}{3\cdot 1} - \frac{x^6}{5\cdot 3\cdot 1} + \cdots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{4\cdot 2} - \frac{x^7}{6\cdot 4\cdot 2} + \cdots\right) \\
&= a_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n-1)!!} + a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!!} \\
&= a_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{\frac{(2n)!}{2^n n!}} + a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n n!} \\
&= a_0\sum_{n=0}^{\infty}(-1)^n\frac{2^n n!}{(2n)!}x^{2n} + a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n n!} \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(1 - \frac{x^2}{1} + \frac{x^4}{3 \cdot 1} - \frac{x^6}{5 \cdot 3 \cdot 1} + \cdots\right)\left(1 - \frac{3x^2}{2} + \frac{5x^4}{4 \cdot 2} - \frac{7x^6}{6 \cdot 4 \cdot 2} + \cdots\right) \\
&\qquad - \left(-\frac{2x}{1} + \frac{4x^3}{3 \cdot 1} - \frac{6x^5}{5 \cdot 3 \cdot 1} + \cdots\right)\left(x - \frac{x^3}{2} + \frac{x^5}{4 \cdot 2} - \frac{x^7}{6 \cdot 4 \cdot 2} + \cdots\right)
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1 - 0 + 0 - \cdots)(1 - 0 + 0 - \cdots) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.