Problem 8

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$xy'' + y' + xy = 0, \qquad x_0 = 1$$

Solution

$x = 1$ is not a zero of the coefficient of y'', so $x = 1$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \to \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \to \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Substitute these series into the ODE.

$$x \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Make it so that $x - 1$ appears rather than x.

$$(x-1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \\ + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + (x-1) \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Bring $x - 1$ into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \\ + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Substitute $k+1=n-1$ in the first sum, $k+1=n-2$ in the second sum, $k+1=n-1$ in the third sum, $k=n$ in the fourth sum, and $k+1=n$ in the fifth sum.

$$\sum_{k+2=2}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^{k+1}+\sum_{k+3=2}^{\infty}(k+3)(k+2)a_{k+3}(x-1)^{k+1}$$
$$+\sum_{k+2=1}^{\infty}(k+2)a_{k+2}(x-1)^{k+1}+\sum_{k=0}^{\infty}a_k(x-1)^{k+1}+\sum_{k+1=0}^{\infty}a_{k+1}(x-1)^{k+1}=0$$

Solve for k.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^{k+1}+\sum_{k=-1}^{\infty}(k+3)(k+2)a_{k+3}(x-1)^{k+1}$$
$$+\sum_{k=-1}^{\infty}(k+2)a_{k+2}(x-1)^{k+1}+\sum_{k=0}^{\infty}a_k(x-1)^{k+1}+\sum_{k=-1}^{\infty}a_{k+1}(x-1)^{k+1}=0$$

Write out the first term of the second, third, and fifth sums.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^{k+1}+2a_2+\sum_{k=0}^{\infty}(k+3)(k+2)a_{k+3}(x-1)^{k+1}$$
$$+a_1+\sum_{k=0}^{\infty}(k+2)a_{k+2}(x-1)^{k+1}+\sum_{k=0}^{\infty}a_k(x-1)^{k+1}+a_0+\sum_{k=0}^{\infty}a_{k+1}(x-1)^{k+1}=0$$

Now that each of the sums has the same limits and factors of $x-1$, they can be combined.

$$(a_0+a_1+2a_2)+\sum_{k=0}^{\infty}\Big[(k+2)(k+1)a_{k+2}(x-1)^{k+1}+(k+3)(k+2)a_{k+3}(x-1)^{k+1}$$
$$+(k+2)a_{k+2}(x-1)^{k+1}+a_k(x-1)^{k+1}+a_{k+1}(x-1)^{k+1}\Big]=0$$

Factor the summand.

$$(a_0+a_1+2a_2)+\sum_{k=0}^{\infty}\{(k+2)[(k+1)a_{k+2}+(k+3)a_{k+3}+a_{k+2}]+a_k+a_{k+1}\}(x-1)^{k+1}=0$$

$$(a_0+a_1+2a_2)+\sum_{k=0}^{\infty}\{(k+2)[(k+2)a_{k+2}+(k+3)a_{k+3}]+a_k+a_{k+1}\}(x-1)^{k+1}=0+0(x-1)+0(x-1)^2+\cdots$$

The coefficients must be zero.

$$a_0+a_1+2a_2=0$$
$$(k+2)[(k+2)a_{k+2}+(k+3)a_{k+3}]+a_k+a_{k+1}=0$$

Solve the previous equations for a_2 and a_{k+3}.

$$a_2=-\frac{a_0}{2}-\frac{a_1}{2}$$
$$a_{k+3}=-\frac{(k+2)^2a_{k+2}+a_{k+1}+a_k}{(k+3)(k+2)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$k=0: \quad a_3 = -\frac{4a_2 + a_1 + a_0}{3\cdot 2} = \frac{a_0}{6} + \frac{a_1}{6}$$
$$k=1: \quad a_4 = -\frac{9a_3 + a_2 + a_1}{4\cdot 3} = -\frac{a_0}{12} - \frac{a_1}{6}$$
$$\vdots$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1(x-1) + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)(x-1)^3 + \left(-\frac{a_0}{12} - \frac{a_1}{6}\right)(x-1)^4 + \cdots \\
&= a_0\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots\right] \\
&\quad + a_1\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots\right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \cdots\right]\left[1 - (x-1) + \frac{1}{2}(x-1)^2 - \frac{2}{3}(x-1)^3 + \cdots\right] \\
&\quad - \left[-(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \cdots\right]\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{6}(x-1)^4 + \cdots\right]
\end{aligned}$$

At $x = 1$ the Wronskian is nonzero,

$$W(y_1, y_2)(1) = (1 - 0 + 0 - \cdots)(1 - 0 + 0 - \cdots) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.