

## Problem 10

In each of Problems 1 through 14:

- Seek power series solutions of the given differential equation about the given point  $x_0$ ; find the recurrence relation.
- Find the first four terms in each of two solutions  $y_1$  and  $y_2$  (unless the series terminates sooner).
- By evaluating the Wronskian  $W(y_1, y_2)(x_0)$ , show that  $y_1$  and  $y_2$  form a fundamental set of solutions.
- If possible, find the general term in each solution.

$$(4 - x^2)y'' + 2y = 0, \quad x_0 = 0$$

### Solution

$x = 0$  is not a zero of the coefficient of  $y''$ , so  $x = 0$  is an ordinary point. As such, the solution for  $y$  can be represented as a power series centered at  $x = 0$ .

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to  $x$  to get  $y'$  and  $y''$ .

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(4 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 4,  $x^2$ , and 2 into the respective summands.

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Start the second sum from  $n = 0$ . This can be done because of the  $n$  and  $n - 1$  factors.

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Substitute  $k = n - 2$  in the first sum and  $k = n$  in the others.

$$\sum_{k+2=2}^{\infty} 4(k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Solve for  $k$ .

$$\sum_{k=0}^{\infty} 4(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_kx^k + \sum_{k=0}^{\infty} 2a_kx^k = 0$$

Now that each of the sums has the same limits and factors of  $x$ , they can be combined.

$$\sum_{k=0}^{\infty} [4(k+2)(k+1)a_{k+2}x^k - k(k-1)a_kx^k + 2a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} [4(k+2)(k+1)a_{k+2} - k(k-1)a_k + 2a_k]x^k = 0$$

$$\sum_{k=0}^{\infty} [4(k+2)(k+1)a_{k+2} - (k^2 - k - 2)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty} [4(k+2)(k+1)a_{k+2} - (k-2)(k+1)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty} (k+1)[4(k+2)a_{k+2} - (k-2)a_k]x^k = 0$$

The coefficients must be zero.

$$4(k+2)a_{k+2} - (k-2)a_k = 0$$

Solve for  $a_{k+2}$ .

$$a_{k+2} = \frac{k-2}{4(k+2)}a_k$$

Plug in enough values of  $k$  to get four terms involving  $a_0$  and four terms involving  $a_1$ .

$$k = 0: \quad a_2 = \frac{-2}{4(2)}a_0 = -\frac{1}{4}a_0$$

$$k = 1: \quad a_3 = \frac{-1}{4(3)}a_1 = \frac{-1}{4 \cdot 3}a_1$$

$$k = 2: \quad a_4 = \frac{0}{4(4)}a_2 = 0$$

$$k = 3: \quad a_5 = \frac{1}{4(5)}a_3 = \frac{1}{4(5)} \left( \frac{-1}{4 \cdot 3}a_1 \right) = \frac{1 \cdot (-1)}{4^2 \cdot 5 \cdot 3}a_1$$

$$k = 4: \quad a_6 = \frac{2}{4(6)}a_4 = 0$$

$$k = 5: \quad a_7 = \frac{3}{4(7)}a_5 = \frac{3}{4(7)} \left[ \frac{1 \cdot (-1)}{4^2 \cdot 5 \cdot 3}a_1 \right] = \frac{3 \cdot 1 \cdot (-1)}{4^3 \cdot 7 \cdot 5 \cdot 3}a_1$$

⋮

Note that  $a_{2k} = 0$  for  $k > 1$ . Therefore,

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 &= \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n \\
 &= \left( a_0 - \frac{1}{4} a_0 x^2 \right) + \left[ a_1 x + \sum_{k=1}^{\infty} \frac{(2k-3)!!(-1)}{4^k (2k+1)!!} x^{2k+1} \right] \\
 &= \left( a_0 - \frac{1}{4} a_0 x^2 \right) + \left[ a_1 x + \sum_{k=1}^{\infty} \frac{(2k-3)!!(-1)}{4^k (2k+1)(2k-1)(2k-3)!!} x^{2k+1} \right] \\
 &= a_0 \left( 1 - \frac{x^2}{4} \right) + a_1 \left[ x - \sum_{k=1}^{\infty} \frac{x^{2k+1}}{4^k (2k+1)(2k-1)} \right] \\
 &= a_0 \left( 1 - \frac{x^2}{4} \right) + a_1 \left( x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \dots \right) \\
 &= a_0 y_1(x) + a_1 y_2(x).
 \end{aligned}$$

Now calculate the Wronskian of  $y_1$  and  $y_2$ .

$$\begin{aligned}
 W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
 &= y_1 y_2' - y_1' y_2 \\
 &= \left( 1 - \frac{x^2}{4} \right) \left[ 1 - \sum_{k=1}^{\infty} \frac{x^{2k}}{4^k (2k-1)} \right] - \left( -\frac{x}{2} \right) \left[ x - \sum_{k=1}^{\infty} \frac{x^{2k+1}}{4^k (2k+1)(2k-1)} \right]
 \end{aligned}$$

At  $x = 0$  the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that  $y_1$  and  $y_2$  form a fundamental set of solutions for the ODE.