Problem 12

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(1-x)y'' + xy' - y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and x into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of the $n-1$ factor, the second sum can be started from $n = 1$. Similarly, because of the n factor, the third sum can be started from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n-2$ in the first sum, $k = n-1$ in the second sum, and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k+1=1}^{\infty} (k+1)k a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty}k(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty}ka_kx^k - \sum_{k=0}^{\infty}a_kx^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k - k(k+1)a_{k+1}x^k + ka_kx^k - a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + ka_k - a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-1)a_k]x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-1)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k(k+1)a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: \quad & a_2 = \frac{0(1)a_1 - (-1)a_0}{(2)(1)} = \frac{a_0}{2\cdot 1} \\
k=1: \quad & a_3 = \frac{1(2)a_2 - 0}{(3)(2)} = \frac{1}{3}\left(\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{3\cdot 2\cdot 1} \\
k=2: \quad & a_4 = \frac{2(3)a_3 - a_2}{(4)(3)} = \frac{2}{4}\left(\frac{a_0}{3\cdot 2\cdot 1}\right) - \frac{1}{4\cdot 3}\left(\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1} \\
& \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \frac{a_0}{2\cdot 1}x^2 + \frac{a_0}{3\cdot 2\cdot 1}x^3 + \frac{a_0}{4\cdot 3\cdot 2\cdot 1}x^4 + \cdots \\
&= a_1 x + a_0\left(1 + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots\right) \\
&= a_1 x + a_0\left(-x + 1 + \frac{x}{1!} + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots\right) \\
&= a_1 x + a_0\left(-x + \sum_{n=0}^{\infty}\frac{x^n}{n!}\right) \\
&= a_1 x + a_0(-x + e^x) \\
&= a_1 y_2(x) + a_0 y_1(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= (-x + e^x)(1) - (-1 + e^x)(x) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.