Problem 13

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$2y'' + xy' + 3y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 2, x, and 3 into the respective summands.

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Because of the factor n, the second sum can be set to start from $n = 0$.

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} 2(k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 3a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} 2(k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 3a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2}x^k + ka_kx^k + 3a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + ka_k + 3a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + (k+3)a_k]x^k = 0$$

The coefficients must be zero.

$$2(k+2)(k+1)a_{k+2} + (k+3)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = -\frac{3}{2(2)(1)}a_0 = -\frac{3\cdot 1}{2^1\cdot 2\cdot 1}a_0 \\
k=1: &\quad a_3 = -\frac{4}{2(3)(2)}a_1 = -\frac{4}{2^1\cdot 3\cdot 2\cdot 1}a_1 = -\frac{4\cdot 2}{2^2\cdot 3\cdot 2\cdot 1}a_1 \\
k=2: &\quad a_4 = -\frac{5}{2(4)(3)}a_2 = -\frac{5}{2(4)(3)}\left[-\frac{3}{2^1\cdot 2\cdot 1}a_0\right] = \frac{5\cdot 3\cdot 1}{2^2\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \\
k=3: &\quad a_5 = -\frac{6}{2(5)(4)}a_3 = -\frac{6}{2(5)(4)}\left[-\frac{4\cdot 2}{2^2\cdot 3\cdot 2\cdot 1}a_1\right] = \frac{6\cdot 4\cdot 2}{2^3\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_1 \\
k=4: &\quad a_6 = -\frac{7}{2(6)(5)}a_4 = -\frac{7}{2(6)(5)}\left(\frac{5\cdot 3\cdot 1}{2^2\cdot 4\cdot 3\cdot 2\cdot 1}a_0\right) = -\frac{7\cdot 5\cdot 3\cdot 1}{2^3\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \\
k=5: &\quad a_7 = -\frac{8}{2(7)(6)}a_5 = -\frac{8}{2(7)(6)}\left(\frac{6\cdot 4\cdot 2}{2^3\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_1\right) = -\frac{8\cdot 6\cdot 4\cdot 2}{2^4\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_1 \\
&\quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n \text{ even}}^{\infty} a_n x^n + \sum_{n \text{ odd}}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{2^k (2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2)!!}{2^{k+1}(2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{\frac{[2(k+1)]!}{2^{k+1}(k+1)!}}{2^k (2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{2^{k+1}(k+1)!}{2^{k+1}(2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2)!}{2^{2k+1}(2k)!(k+1)!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)!}{(2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2)(2k+1)(2k)!}{2^{2k+1}(2k)!(k+1)k!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)!}{(2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{2^{2k} k!} x^{2k} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)!}{(2k+1)!} x^{2k+1} \\
&= a_0 \left(1 - \frac{3x^2}{4} + \frac{5x^4}{32} - \frac{7x^6}{384} + \cdots\right) + a_1 \left(x - \frac{x^3}{3} + \frac{x^5}{20} - \frac{x^7}{210} + \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(1 - \frac{3x^2}{4} + \frac{5x^4}{32} - \frac{7x^6}{384} + \cdots\right)\left(1 - x^2 + \frac{x^4}{4} - \frac{x^6}{30} + \cdots\right) \\
&\qquad - \left(-\frac{3x}{2} + \frac{5x^3}{8} - \frac{7x^5}{64} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{20} - \frac{x^7}{210} + \cdots\right)
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.