Problem 14

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$2y'' + (x+1)y' + 3y = 0, \qquad x_0 = 2$$

Solution

$x = 2$ is not a zero of the coefficient of y'', so $x = 2$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 2$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n \quad \to \quad y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \quad \to \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

Substitute these series into the ODE.

$$2\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} + (x+1)\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 3\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

Make it so that the factor in front of the second sum is $x-2$.

$$2\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} + (x-2)\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 3\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 3\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

Bring 2, $x-2$, 3, and 3 into the respective summands.

$$\sum_{n=2}^{\infty} 2n(n-1) a_n (x-2)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-2)^n + \sum_{n=1}^{\infty} 3n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} 3a_n (x-2)^n = 0$$

Because of the factor of n, the second sum can be set to start from $n = 0$.

$$\sum_{n=2}^{\infty} 2n(n-1) a_n (x-2)^{n-2} + \sum_{n=0}^{\infty} n a_n (x-2)^n + \sum_{n=1}^{\infty} 3n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} 3a_n (x-2)^n = 0$$

Substitute $k = n-2$ in the first sum, $k = n-1$ in the third sum, and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} 2(k+2)(k+1) a_{k+2} (x-2)^k + \sum_{k=0}^{\infty} k a_k (x-2)^k + \sum_{k+1=1}^{\infty} 3(k+1) a_{k+1} (x-2)^k + \sum_{k=0}^{\infty} 3a_k (x-2)^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}(x-2)^k + \sum_{k=0}^{\infty} ka_k(x-2)^k + \sum_{k=0}^{\infty} 3(k+1)a_{k+1}(x-2)^k + \sum_{k=0}^{\infty} 3a_k(x-2)^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2}(x-2)^k + ka_k(x-2)^k + 3(k+1)a_{k+1}(x-2)^k + 3a_k(x-2)^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + ka_k + 3(k+1)a_{k+1} + 3a_k](x-2)^k = 0$$

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + 3(k+1)a_{k+1} + (k+3)a_k](x-2)^k = 0$$

The coefficients must be zero.

$$2(k+2)(k+1)a_{k+2} + 3(k+1)a_{k+1} + (k+3)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = -\frac{3(k+1)a_{k+1} + (k+3)a_k}{2(k+2)(k+1)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = -\frac{3(1)a_1 + 3a_0}{2(2)(1)} = -\frac{3}{4}a_0 - \frac{3}{4}a_1 \\
k=1: &\quad a_3 = -\frac{3(2)a_2 + 4a_1}{2(3)(2)} = \frac{3}{8}a_0 + \frac{1}{24}a_1 \\
k=2: &\quad a_4 = -\frac{3(3)a_3 + 5a_2}{2(4)(3)} = \frac{1}{64}a_0 + \frac{9}{64}a_1 \\
&\quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n(x-2)^n \\
&= a_0 + a_1(x-2) + \left(-\frac{3}{4}a_0 - \frac{3}{4}a_1\right)(x-2)^2 + \left(\frac{3}{8}a_0 + \frac{1}{24}a_1\right)(x-2)^3 + \left(\frac{1}{64}a_0 + \frac{9}{64}a_1\right)(x-2)^4 + \cdots \\
&= a_0\left[1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \cdots\right] \\
&\qquad + a_1\left[(x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \cdots\right] \\
&= a_0y_1(x) + a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left[1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \cdots\right]\left[1 - \frac{3}{2}(x-2) + \frac{1}{8}(x-2)^2 + \frac{9}{16}(x-2)^3 + \cdots\right] \\ &\quad - \left[-\frac{3}{2}(x-2) + \frac{9}{8}(x-2)^2 + \frac{1}{16}(x-2)^3 + \cdots\right] \\ &\qquad \times \left[(x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \cdots\right] \end{aligned}$$

At $x = 2$ the Wronskian is nonzero,

$$W(y_1, y_2)(2) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.