Problem 15

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

$$y'' - xy' - y = 0, \qquad y(0) = 2, \quad y'(0) = 1; \qquad \text{see Problem 2}$$

Solution

The initial conditions are provided at $x = 0$, which is not a zero of the coefficient of y''; thus, it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n$ in the second sum, and $k = n$ in the third sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

Bring x inside the summand and start the second series from $k = 0$.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Since the three sums have the same limits and x^k, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k - k a_k x^k - a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - (k+1) a_k] x^k = 0$$

$$\sum_{k=0}^{\infty}(k+1)[(k+2)a_{k+2}-a_k]x^k = 0$$

The quantity in square brackets must be zero.

$$(k+2)a_{k+2}-a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{a_k}{k+2}$$

Plug in different values of k to determine a pattern for a_k.

$$\begin{aligned}
a_2 &= \frac{a_0}{2} & a_3 &= \frac{a_1}{3} \\
a_4 &= \frac{a_2}{4} = \frac{a_0}{4\cdot 2} & a_5 &= \frac{a_3}{5} = \frac{a_1}{5\cdot 3} \\
a_6 &= \frac{a_4}{6} = \frac{a_0}{6\cdot 4\cdot 2} & a_7 &= \frac{a_5}{7} = \frac{a_1}{7\cdot 5\cdot 3} \\
&\vdots & &\vdots \\
a_{2k} &= \frac{a_0}{(2k)!!} = \frac{a_0}{2^k k!} & a_{2k+1} &= \frac{a_1}{(2k+1)!!} = \frac{a_1}{\frac{(2k+2)!}{2^{k+1}(k+1)!}} = \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1
\end{aligned}$$

As a result, the general solution is

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n\text{ even}}^{\infty} a_n x^n + \sum_{n\text{ odd}}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{a_0}{2^k k!}x^{2k} + \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1 x^{2k+1} \\
&= a_0\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!}x^{2k+1} \\
&= a_0\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+\cdots\right) + a_1\left(x+\frac{x^3}{3}+\frac{x^5}{15}+\frac{x^7}{105}+\cdots\right).
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = a_0\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2^{k-1}(k-1)!} + a_1\sum_{k=0}^{\infty}\frac{2^{k+1}(k+1)!}{(2k+2)!}(2k+1)x^{2k}$$

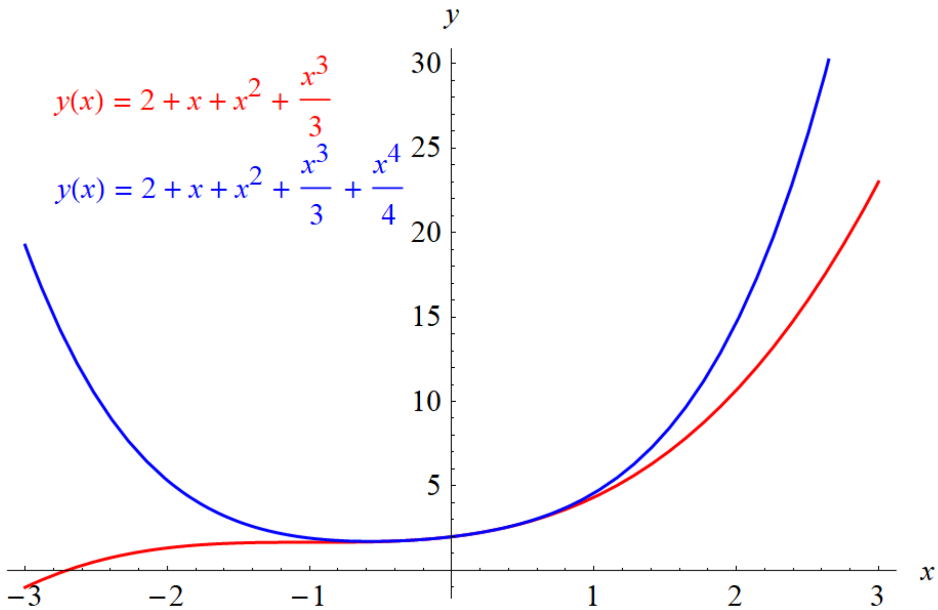
Apply the initial conditions now to determine a_0 and a_1.

$$\begin{aligned}
y(0) &= a_0 = 2 \\
y'(0) &= a_1 = 1
\end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= 2\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots\right) + \left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots\right) \\ &= 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \cdots . \end{aligned}$$

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.



The interval in which the four-term approximation is accurate is roughly $-0.75 < x < 0.75$—the two curves start to deviate outside of it.