Problem 17

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

$$y'' + xy' + 2y = 0, \qquad y(0) = 4, \quad y'(0) = -1; \qquad \text{see Problem 7}$$

Solution

The initial conditions are provided at $x = 0$, which is not a zero of the coefficient of y''; thus, it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and 2 into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the other sums.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Solve for k in the first sum. The second sum can be set to start from $k = 0$ because k is present in the summand.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k + k a_k x^k + 2 a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}+ka_k+2a_k]x^k=0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}+(k+2)a_k]x^k=0$$

$$\sum_{k=0}^{\infty}(k+2)[(k+1)a_{k+2}+a_k]x^k=0$$

The coefficients must be zero.

$$(k+1)a_{k+2}+a_k=0$$

Solve for a_{k+2}.

$$a_{k+2}=-\frac{a_k}{k+1}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2=-\frac{a_0}{1}\\
k=1: &\quad a_3=-\frac{a_1}{2}\\
k=2: &\quad a_4=-\frac{a_2}{3}=\frac{a_0}{3\cdot 1}\\
k=3: &\quad a_5=-\frac{a_3}{4}=\frac{a_1}{4\cdot 2}\\
k=4: &\quad a_6=-\frac{a_4}{5}=-\frac{a_0}{5\cdot 3\cdot 1}\\
k=5: &\quad a_7=-\frac{a_5}{6}=-\frac{a_1}{6\cdot 4\cdot 2}\\
&\vdots
\end{aligned}$$

As a result, the general solution is

$$\begin{aligned}
y(x)&=\sum_{n=0}^{\infty}a_nx^n\\
&=a_0+a_1x-\frac{a_0}{1}x^2-\frac{a_1}{2}x^3+\frac{a_0}{3\cdot 1}x^4+\frac{a_1}{4\cdot 2}x^5-\frac{a_0}{5\cdot 3\cdot 1}x^6-\frac{a_1}{6\cdot 4\cdot 2}x^7+\cdots\\
&=a_0\left(1-\frac{x^2}{1}+\frac{x^4}{3\cdot 1}-\frac{x^6}{5\cdot 3\cdot 1}+\cdots\right)+a_1\left(x-\frac{x^3}{2}+\frac{x^5}{4\cdot 2}-\frac{x^7}{6\cdot 4\cdot 2}+\cdots\right)\\
&=a_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n-1)!!}+a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!!}\\
&=a_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{\frac{(2n)!}{2^nn!}}+a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^nn!}\\
&=a_0\sum_{n=0}^{\infty}(-1)^n\frac{2^nn!}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^nn!}.
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x)=a_0\sum_{n=1}^{\infty}(-1)^n\frac{2^nn!}{(2n)!}(2n)x^{2n-1}+a_1\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{2^nn!}$$

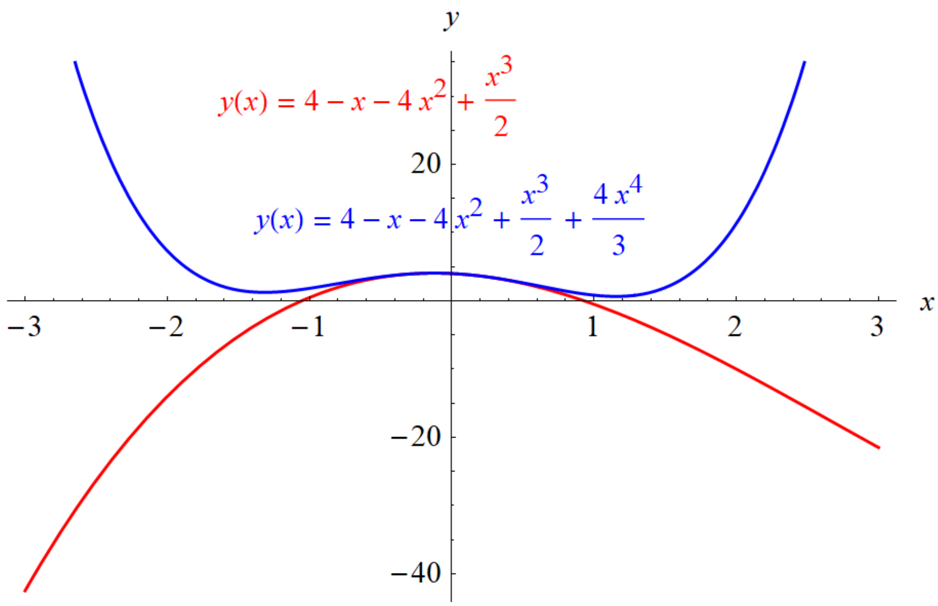
Apply the initial conditions now to determine a_0 and a_1.

$$y(0) = a_0 = 4$$
$$y'(0) = a_1 = -1$$

Therefore,

$$\begin{aligned} y(x) &= 4\sum_{n=0}^{\infty}(-1)^n \frac{2^n n!}{(2n)!}x^{2n} - \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^n n!} \\ &= 4\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + \cdots\right) - \left(x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \cdots\right) \\ &= 4 - x - 4x^2 + \frac{x^3}{2} + \frac{4x^4}{3} - \cdots . \end{aligned}$$

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.



The interval in which the four-term approximation is accurate is roughly $-0.5 < x < 0.5$—the two curves start to deviate outside of it.