Problem 19

(a) By making the change of variable $x - 1 = t$ and assuming that y has a Taylor series in powers of t, find two series solutions of

$$y'' + (x-1)^2 y' + (x^2 - 1)y = 0$$

in powers of $x - 1$.

(b) Show that you obtain the same result by assuming that y has a Taylor series in powers of $x - 1$ and also expressing the coefficient $x^2 - 1$ in powers of $x - 1$.

Solution

Part (a)

Rewrite the ODE.

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} + (x-1)(x+1)y = 0$$

Make the change of variables, $t = x - 1$.

$$\frac{d^2y}{dx^2} + t^2\frac{dy}{dx} + t(t+2)y = 0$$

The aim now is to find what dy/dx and d^2y/dx^2 are in terms of this new variable. Use the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}(1) = \frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dt}\right) = (1)\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

As a result of changing variables, the new ODE is

$$\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + t(t+2)y = 0.$$

Assume that y has a Taylor series in powers of t.

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Differentiate this series twice with respect to t to get dy/dt and d^2y/dt^2.

$$y = \sum_{n=0}^{\infty} a_n t^n \quad \rightarrow \quad \frac{dy}{dt} = \sum_{n=1}^{\infty} a_n n t^{n-1} \quad \rightarrow \quad \frac{d^2y}{dt^2} = \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^2\sum_{n=1}^{\infty} a_n n t^{n-1} + t(t+2)\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^2 \sum_{n=1}^{\infty} a_n n t^{n-1} + (t^2+2t)\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^2 \sum_{n=1}^{\infty} a_n n t^{n-1} + t^2\sum_{n=0}^{\infty} a_n t^n + 2t\sum_{n=0}^{\infty} a_n t^n = 0$$

Bring t^2, t^2, and $2t$ into the respective summands.

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + \sum_{n=1}^{\infty} a_n n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2a_n t^{n+1} = 0$$

Substitute $k+2=n-2$ in the first sum, $k+2=n+1$ in the second sum, $k=n$ in the third sum, and $k+2=n+1$ in the fourth sum.

$$\sum_{k+4=2}^{\infty} a_{k+4}(k+4)(k+3)t^{k+2} + \sum_{k+1=1}^{\infty} a_{k+1}(k+1)t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + \sum_{k+1=0}^{\infty} 2a_{k+1}t^{k+2} = 0$$

Solve for k.

$$\sum_{k=-2}^{\infty} (k+4)(k+3)a_{k+4}t^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1}t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + \sum_{k=-1}^{\infty} 2a_{k+1}t^{k+2} = 0$$

Write out the first two terms of the first sum and the first term of the fourth sum.

$$2a_2 + 6a_3 t + \sum_{k=0}^{\infty} (k+4)(k+3)a_{k+4}t^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1}t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + 2a_0 t + \sum_{k=0}^{\infty} 2a_{k+1}t^{k+2} = 0$$

Now that each of the sums has the same limits and factors of t, they can be combined.

$$2a_2 + (2a_0+6a_3)t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4}t^{k+2} + (k+1)a_{k+1}t^{k+2} + a_k t^{k+2} + 2a_{k+1}t^{k+2}] = 0$$

Factor the summand.

$$2a_2 + (2a_0+6a_3)t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4} + (k+1)a_{k+1} + a_k + 2a_{k+1}]t^{k+2} = 0$$

$$2a_2 + (2a_0+6a_3)t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4} + (k+3)a_{k+1} + a_k]t^{k+2} = 0 + 0t + 0t^2 + \cdots$$

Match the coefficients on both sides.

$$2a_2 = 0$$
$$2a_0 + 6a_3 = 0$$
$$(k+4)(k+3)a_{k+4} + (k+3)a_{k+1} + a_k = 0$$

Solve for a_2, a_3, and a_{k+4}.

$$a_2 = 0$$
$$a_3 = -\frac{a_0}{3}$$
$$a_{k+4} = -\frac{(k+3)a_{k+1} + a_k}{(k+4)(k+3)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_4 = -\frac{3a_1+a_0}{4\cdot 3} = -\frac{a_0}{12} - \frac{a_1}{4} \\
k=1: &\quad a_5 = -\frac{4a_2+a_1}{5\cdot 4} = -\frac{a_1}{20} \\
k=2: &\quad a_6 = -\frac{5a_3+a_2}{6\cdot 5} = \frac{a_0}{18} \\
k=3: &\quad a_7 = -\frac{6a_4+a_3}{7\cdot 6} = \frac{a_0}{252} + \frac{a_1}{28} \\
&\quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \\
&= a_0 + a_1 t - \frac{a_0}{3}t^3 + \left(-\frac{a_0}{12} - \frac{a_1}{4}\right)t^4 - \frac{a_1}{20}t^5 + \frac{a_0}{18}t^6 + \left(\frac{a_0}{252} + \frac{a_1}{28}\right)t^7 + \cdots \\
&= a_0\left(1 - \frac{t^3}{3} - \frac{t^4}{12} + \frac{t^6}{18} + \cdots\right) + a_1\left(t - \frac{t^4}{4} - \frac{t^5}{20} + \frac{t^7}{28} + \cdots\right).
\end{aligned}$$

Changing back to the original variable x, we have

$$\begin{aligned}
y(x) &= a_0\left[1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + \cdots\right] \\
&\qquad + a_1\left[(x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{28}(x-1)^7 + \cdots\right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Part (b)

Rewrite the ODE.

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} + (x-1)(x+1)y = 0$$

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} + (x-1)(x-1+2)y = 0$$

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} + (x-1)^2y + 2(x-1)y = 0$$

Assume that y has a Taylor series expansion in powers of $x-1$.

$$y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$$

Differentiate this series twice with respect to x to get dy/dx and d^2y/dt^2.

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n \quad \rightarrow \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \quad \rightarrow \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

Substitute these expressions into the ODE.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (x-1)^2\sum_{n=1}^{\infty} na_n(x-1)^{n-1} + (x-1)^2\sum_{n=0}^{\infty} a_n(x-1)^n + 2(x-1)\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Bring $(x-1)^2$, $(x-1)^2$, and $2(x-1)$ into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^{n+2} + \sum_{n=0}^{\infty} 2a_n(x-1)^{n+1} = 0$$

Substitute $k+2 = n-2$ in the first sum, $k+2 = n+1$ in the second sum, $k = n$ in the third sum, and $k+2 = n+1$ in the fourth sum.

$$\sum_{k+4=2}^{\infty} a_{k+4}(k+4)(k+3)(x-1)^{k+2} + \sum_{k+1=1}^{\infty} a_{k+1}(k+1)(x-1)^{k+2} + \sum_{k=0}^{\infty} a_k(x-1)^{k+2} + \sum_{k+1=0}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0$$

Solve for k.

$$\sum_{k=-2}^{\infty} (k+4)(k+3)a_{k+4}(x-1)^{k+2} + \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^{k+2} + \sum_{k=0}^{\infty} a_k(x-1)^{k+2} + \sum_{k=-1}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0$$

Write out the first two terms of the first sum and the first term of the fourth sum.

$$2a_2 + 6a_3(x-1) + \sum_{k=0}^{\infty}(k+4)(k+3)a_{k+4}(x-1)^{k+2} + \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^{k+2} + \sum_{k=0}^{\infty} a_k(x-1)^{k+2} + 2a_0(x-1) + \sum_{k=0}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0$$

Now that each of the sums has the same limits and factors of $(x-1)$, they can be combined.

$$2a_2 + (2a_0 + 6a_3)(x-1)$$
$$+\sum_{k=0}^{\infty}[(k+4)(k+3)a_{k+4}(x-1)^{k+2} + (k+1)a_{k+1}(x-1)^{k+2} + a_k(x-1)^{k+2} + 2a_{k+1}(x-1)^{k+2}] = 0$$

Factor the summand.

$$2a_2 + (2a_0 + 6a_3)(x-1) + \sum_{k=0}^{\infty}[(k+4)(k+3)a_{k+4} + (k+1)a_{k+1} + a_k + 2a_{k+1}](x-1)^{k+2} = 0$$

$$2a_2 + (2a_0 + 6a_3)(x-1) + \sum_{k=0}^{\infty}[(k+4)(k+3)a_{k+4} + (k+3)a_{k+1} + a_k](x-1)^{k+2}$$
$$= 0 + 0(x-1) + 0(x-1)^2 + \cdots$$

Match the coefficients on both sides.

$$\begin{aligned} 2a_2 &= 0 \\ 2a_0 + 6a_3 &= 0 \\ (k+4)(k+3)a_{k+4} + (k+3)a_{k+1} + a_k &= 0 \end{aligned}$$

Solve for a_2, a_3, and a_{k+4}.

$$\begin{aligned} a_2 &= 0 \\ a_3 &= -\frac{a_0}{3} \\ a_{k+4} &= -\frac{(k+3)a_{k+1} + a_k}{(k+4)(k+3)} \end{aligned}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned} k=0: &\quad a_4 = -\frac{3a_1 + a_0}{4\cdot 3} = -\frac{a_0}{12} - \frac{a_1}{4} \\ k=1: &\quad a_5 = -\frac{4a_2 + a_1}{5\cdot 4} = -\frac{a_1}{20} \\ k=2: &\quad a_6 = -\frac{5a_3 + a_2}{6\cdot 5} = \frac{a_0}{18} \\ k=3: &\quad a_7 = -\frac{6a_4 + a_3}{7\cdot 6} = \frac{a_0}{252} + \frac{a_1}{28} \\ &\quad \vdots \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots \\ &= a_0 + a_1(x-1) - \frac{a_0}{3}(x-1)^3 + \left(-\frac{a_0}{12} - \frac{a_1}{4}\right)(x-1)^4 - \frac{a_1}{20}(x-1)^5 + \frac{a_0}{18}(x-1)^6 + \left(\frac{a_0}{252} + \frac{a_1}{28}\right)(x-1)^7 + \cdot \\ &= a_0\left[1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + \cdots\right] \\ &\qquad + a_1\left[(x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{28}(x-1)^7 + \cdots\right]. \end{aligned}$$