Problem 22

Consider the initial value problem $y' = \sqrt{1-y^2}$, $y(0) = 0$.

(a) Show that $y = \sin x$ is the solution of this initial value problem.

(b) Look for a solution of the initial value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

Solution

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

Separate variables.

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

Integrate both sides.

$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx$$

To evaluate the integral on the left, make the substitution $y = \sin\theta$. Then $dy = \cos\theta\, d\theta$.

$$\int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int dx$$

$$\int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} = \int dx$$

$$\int \frac{\cos\theta\, d\theta}{\cos\theta} = \int dx$$

$$\int d\theta = \int dx$$

$$\theta = x + C$$

$$\sin^{-1} y = x + C$$

Apply the initial condition $y(0) = 0$ to determine C.

$$0 = C$$

Therefore,

$$\sin^{-1} y = x$$

$$y(x) = \sin x.$$

The series solution for the ODE is just the Taylor series expansion of $\sin x$ about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \cdots$$