Problem 27

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

$$y'' + x^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 0; \qquad \text{see Problem 4}$$

Solution

In Problem 4 the general solution was found to be (setting $k = 1$)

$$\begin{aligned} y(x) &= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1}{4}\right)^{2m} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)} x^{4m} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1}{4}\right)^{2m} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(m+\frac{5}{4}\right)} x^{4m+1} \\ &= a_0 \left(1 - \frac{1}{4\cdot 3}x^4 + \frac{1}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{1}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \frac{1}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{16} - \cdots\right) \\ &\quad + a_1 \left(x - \frac{1}{5\cdot 4}x^5 + \frac{1}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}x^{13} + \frac{1}{17\cdot 16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}x^{17} - \cdots\right). \end{aligned}$$

Differentiate it with respect to x.

$$\begin{aligned} y'(x) = a_0 &\left(-\frac{1}{3}x^3 + \frac{1}{7\cdot 4\cdot 3}x^7 - \frac{1}{11\cdot 8\cdot 7\cdot 4\cdot 3}x^{11} + \frac{1}{15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{15} - \cdots\right) \\ &+ a_1 \left(1 - \frac{1}{4}x^4 + \frac{1}{8\cdot 5\cdot 4}x^8 - \frac{1}{12\cdot 9\cdot 8\cdot 5\cdot 4}x^{12} + \frac{1}{16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}x^{16} - \cdots\right) \end{aligned}$$

Now apply the initial conditions, $y(0) = 1$ and $y'(0) = 0$, to determine a_0 and a_1.

$$\begin{aligned} y(0) &= a_0 = 1 \\ y'(0) &= a_1 = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1}{4}\right)^{2m} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(m+\frac{3}{4}\right)} x^{4m} \\ &= 1 - \frac{1}{4\cdot 3}x^4 + \frac{1}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{1}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \frac{1}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{16} - \cdots \\ &= 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88\,704}x^{12} + \frac{1}{21\,288\,960}x^{16} - \cdots. \end{aligned}$$

Below is a plot of the various partial sums versus x.