Problem 10

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(4 - x^2)y'' + 2y = 0, \qquad x_0 = 0$$

Solution

$x_0 = 0$ is an ordinary point, so the solution can be represented as a power series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(4 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 4, x^2, and 2 into the respective summands.

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Start the second sum from $n = 0$. This can be done because of the n and $n - 1$ factors.

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} 4(k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} 4(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[4(k+2)(k+1)a_{k+2}x^k - k(k-1)a_k x^k + 2a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[4(k+2)(k+1)a_{k+2} - k(k-1)a_k + 2a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[4(k+2)(k+1)a_{k+2} - (k^2-k-2)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[4(k+2)(k+1)a_{k+2} - (k-2)(k+1)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}(k+1)[4(k+2)a_{k+2} - (k-2)a_k]x^k = 0$$

The coefficients must be zero.

$$4(k+2)a_{k+2} - (k-2)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k-2}{4(k+2)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = \frac{-2}{4(2)}a_0 = -\frac{1}{4}a_0 \\
k=1: &\quad a_3 = \frac{-1}{4(3)}a_1 = \frac{-1}{4\cdot 3}a_1 \\
k=2: &\quad a_4 = \frac{0}{4(4)}a_2 = 0 \\
k=3: &\quad a_5 = \frac{1}{4(5)}a_3 = \frac{1}{4(5)}\left(\frac{-1}{4\cdot 3}a_1\right) = \frac{1\cdot(-1)}{4^2\cdot 5\cdot 3}a_1 \\
k=4: &\quad a_6 = \frac{2}{4(6)}a_4 = 0 \\
k=5: &\quad a_7 = \frac{3}{4(7)}a_5 = \frac{3}{4(7)}\left[\frac{1\cdot(-1)}{4^2\cdot 5\cdot 3}a_1\right] = \frac{3\cdot 1\cdot(-1)}{4^3\cdot 7\cdot 5\cdot 3}a_1 \\
&\quad \vdots
\end{aligned}$$

Note that $a_{2k} = 0$ for $k > 1$. Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n \\
&= \left(a_0 - \frac{1}{4}a_0 x^2\right) + \left[a_1 x + \sum_{k=1}^{\infty} \frac{(2k-3)!!(-1)}{4^k (2k+1)!!} x^{2k+1}\right] \\
&= \left(a_0 - \frac{1}{4}a_0 x^2\right) + \left[a_1 x + \sum_{k=1}^{\infty} \frac{(2k-3)!!(-1)}{4^k (2k+1)(2k-1)(2k-3)!!} x^{2k+1}\right] \\
&= a_0\left(1 - \frac{x^2}{4}\right) + a_1\left[x - \sum_{k=1}^{\infty} \frac{x^{2k+1}}{4^k (2k+1)(2k-1)}\right] \\
&= a_0\left(1 - \frac{x^2}{4}\right) + a_1\left(x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(1 - \frac{x^2}{4}\right)\left[1 - \sum_{k=1}^{\infty} \frac{x^{2k}}{4^k (2k-1)}\right] - \left(-\frac{x}{2}\right)\left[x - \sum_{k=1}^{\infty} \frac{x^{2k+1}}{4^k (2k+1)(2k-1)}\right]
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.