Problem 16

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

$$(2+x^2)y'' - xy' + 4y = 0, \qquad y(0) = -1, \quad y'(0) = 3; \qquad \text{see Problem 6}$$

Solution

The initial conditions are provided at $x = 0$, which is not a zero of the coefficient of y''; thus, it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 2, x^2, x, and 4 into the respective summands.

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the other sums.

$$\sum_{k+2=2}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$$

Solve for k in the first sum. The second and third sums can be set to start from $k = 0$ because of the k and $k - 1$ factors.

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$$

Since each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty} [2(k+2)(k+1)a_{k+2}x^k + k(k-1)a_k x^k - k a_k x^k + 4a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2}+k(k-1)a_k-ka_k+4a_k]x^k=0$$

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2}+(k^2-2k+4)a_k]x^k=0$$

The coefficients must be zero.

$$2(k+2)(k+1)a_{k+2}+(k^2-2k+4)a_k=0$$

Solve for a_{k+2}.

$$a_{k+2}=-\frac{k^2-2k+4}{2(k+2)(k+1)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2=-a_0\\
k=1: &\quad a_3=-\frac{1}{4}a_1\\
k=2: &\quad a_4=-\frac{1}{6}a_2=\frac{1}{6}a_0\\
k=3: &\quad a_5=-\frac{7}{40}a_3=\frac{7}{160}a_1\\
k=4: &\quad a_6=-\frac{1}{5}a_4=-\frac{1}{30}a_0\\
k=5: &\quad a_7=-\frac{19}{84}a_5=-\frac{19}{1920}a_1\\
k=6: &\quad a_8=-\frac{1}{4}a_6=\frac{1}{120}a_0\\
k=7: &\quad a_9=-\frac{13}{48}a_7=\frac{247}{92\,160}a_1\\
&\quad\vdots
\end{aligned}$$

As a result, the general solution is

$$\begin{aligned}
y(x)&=\sum_{n=0}^{\infty}a_nx^n\\
&=a_0+a_1x-a_0x^2-\frac{1}{4}a_1x^3+\frac{1}{6}a_0x^4+\frac{7}{160}a_1x^5-\frac{1}{30}a_0x^6-\frac{19}{1920}a_1x^7+\frac{1}{120}a_0x^8+\frac{247}{92\,160}a_1x^9+\cdots\\
&=a_0\left(1-x^2+\frac{x^4}{6}-\frac{x^6}{30}+\cdots\right)+a_1\left(x-\frac{x^3}{4}+\frac{7x^5}{160}-\frac{19x^7}{1920}+\cdots\right).
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x)=a_0\left(-2x+\frac{2x^3}{3}-\frac{x^5}{5}+\cdots\right)+a_1\left(1-\frac{3x^2}{4}+\frac{7x^4}{32}-\frac{133}{1920}x^6+\cdots\right)$$

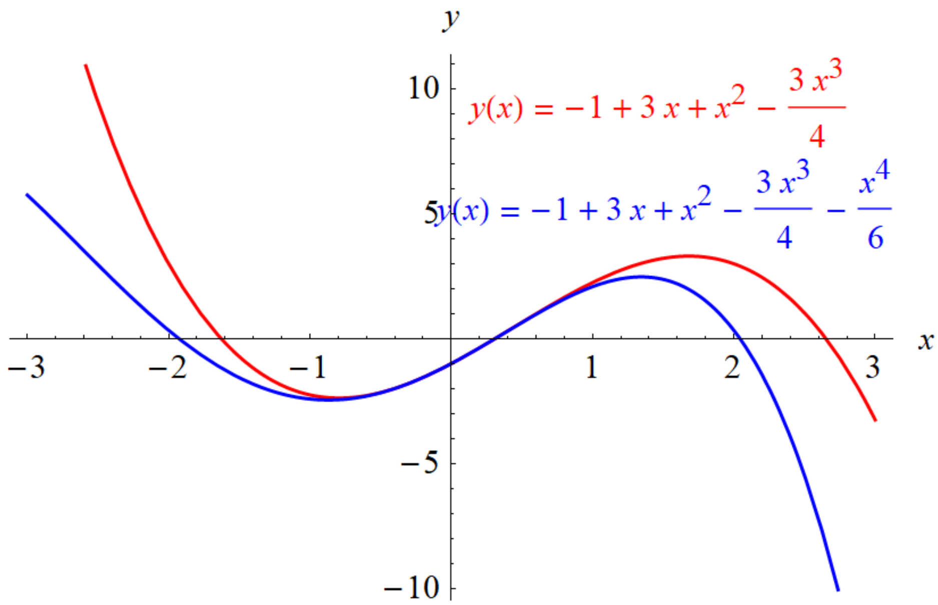
Apply the initial conditions now to determine a_0 and a_1.

$$\begin{aligned}
y(0)&=a_0=-1\\
y'(0)&=a_1=3
\end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= -\left(1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots\right) + 3\left(x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots\right) \\ &= -1 + 3x + x^2 - \frac{3x^3}{4} - \frac{x^4}{6} + \cdots . \end{aligned}$$

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.



The interval in which the four-term approximation is accurate is roughly $-0.9 < x < 0.9$—the two curves start to deviate outside of it.