Problem 18

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

$$(1-x)y'' + xy' - y = 0, \qquad y(0) = -3, \quad y'(0) = 2; \qquad \text{see Problem 12}$$

Solution

The initial conditions are provided at $x = 0$, which is not a zero of the coefficient of y''; thus, it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and x into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of the $n-1$ factor, the second sum can be started from $n = 1$. Similarly, because of the n factor, the third sum can be started from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n-2$ in the first sum, $k = n-1$ in the second sum, and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k+1=1}^{\infty} (k+1)k a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k - k(k+1)a_{k+1}x^k + ka_kx^k - a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + ka_k - a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-1)a_k]x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-1)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k(k+1)a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = \frac{0(1)a_1 - (-1)a_0}{(2)(1)} = \frac{a_0}{2\cdot 1} \\
k=1: &\quad a_3 = \frac{1(2)a_2 - 0}{(3)(2)} = \frac{1}{3}\left(\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{3\cdot 2\cdot 1} \\
k=2: &\quad a_4 = \frac{2(3)a_3 - a_2}{(4)(3)} = \frac{2}{4}\left(\frac{a_0}{3\cdot 2\cdot 1}\right) - \frac{1}{4\cdot 3}\left(\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1} \\
&\quad \vdots
\end{aligned}$$

As a result, the general solution is

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1x + \frac{a_0}{2\cdot 1}x^2 + \frac{a_0}{3\cdot 2\cdot 1}x^3 + \frac{a_0}{4\cdot 3\cdot 2\cdot 1}x^4 + \cdots \\
&= a_1x + a_0\left(1 + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots\right) \\
&= a_1x + a_0\left(-x + 1 + \frac{x}{1!} + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots\right) \\
&= a_1x + a_0\left(-x + \sum_{n=0}^{\infty}\frac{x^n}{n!}\right) \\
&= a_1x + a_0(-x + e^x).
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = a_1 + a_0(-1 + e^x)$$

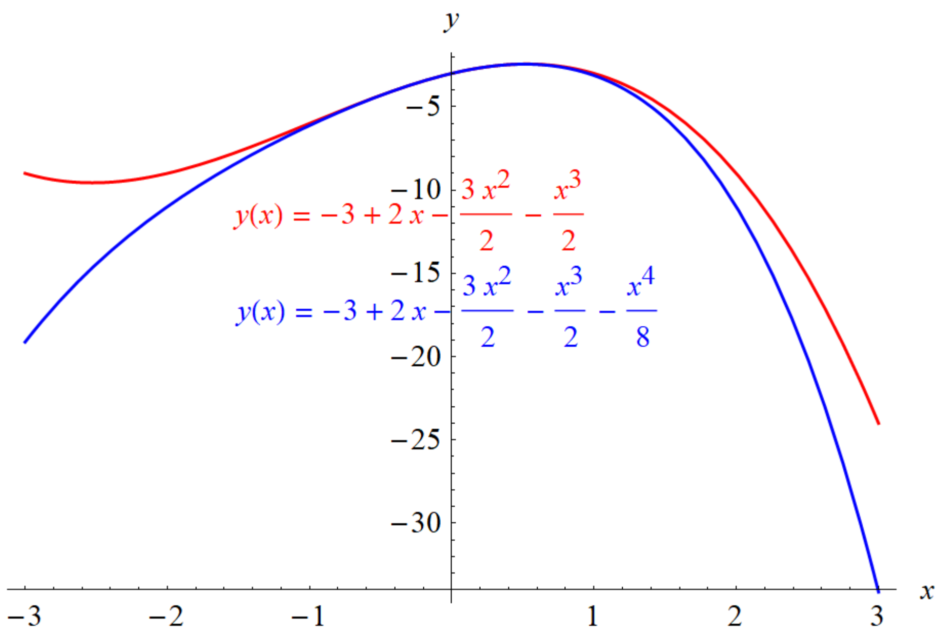
Apply the initial conditions now to determine a_0 and a_1.

$$y(0) = a_0 = -3$$
$$y'(0) = a_1 = 2$$

Therefore,

$$\begin{aligned} y(x) &= 2x - 3(-x + e^x) \\ &= 5x - 3e^x \\ &= 5x - 3\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \\ &= -3 + 2x - \frac{3x^2}{2} - \frac{x^3}{2} - \frac{x^4}{8} - \cdots . \end{aligned}$$

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.



The interval in which the four-term approximation is accurate is roughly $-1 < x < 1$—the two curves start to deviate outside of it.