Problem 2

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' - xy' - y = 0, \qquad x_0 = 0$$

Solution

$x_0 = 0$ is an ordinary point, so the solution can be represented as a power series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n$ in the second sum, and $k = n$ in the third sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

Bring x inside the summand and start the second series from $k = 0$.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Since the three sums have the same limits and x^k, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k - k a_k x^k - a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - (k+1)a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}(k+1)[(k+2)a_{k+2} - a_k]x^k = 0$$

The quantity in square brackets must be zero.

$$(k+2)a_{k+2} - a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{a_k}{k+2}$$

Plug in different values of k to determine a pattern for a_k.

$$\begin{aligned}
a_2 &= \frac{a_0}{2} & a_3 &= \frac{a_1}{3} \\
a_4 &= \frac{a_2}{4} = \frac{a_0}{4\cdot 2} & a_5 &= \frac{a_3}{5} = \frac{a_1}{5\cdot 3} \\
a_6 &= \frac{a_4}{6} = \frac{a_0}{6\cdot 4\cdot 2} & a_7 &= \frac{a_5}{7} = \frac{a_1}{7\cdot 5\cdot 3} \\
&\vdots & &\vdots \\
a_{2k} &= \frac{a_0}{(2k)!!} = \frac{a_0}{2^k k!} & a_{2k+1} &= \frac{a_1}{(2k+1)!!} = \frac{a_1}{\frac{(2k+2)!}{2^{k+1}(k+1)!}} = \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n \text{ even}}^{\infty} a_n x^n + \sum_{n \text{ odd}}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{a_0}{2^k k!}x^{2k} + \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1 x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!}x^{2k+1} \\
&= a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right]' - \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right)' \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right] \\ &= \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} (2k+1) x^{2k}\right] - \left(\sum_{k=1}^{\infty} 2k \frac{x^{2k-1}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right] \end{aligned}$$

At $x = 0$ the Wronskian is

$$\begin{aligned} W(y_1, y_2)(0) &= \left(\frac{1}{2^0 0!} + 0 + 0 + \cdots\right) \left[\frac{2^1 1!}{(2!)}(1) + 0 + 0 + \cdots\right] - (0 + 0 + \cdots)\,[0 + 0 + \cdots] \\ &= (1)(1) \\ &= 1. \end{aligned}$$

Since the Wronskian is not zero at $x = 0$, the two functions, y_1 and y_2, form a fundamental set of solutions for the ODE.