Problem 25

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

$$y'' + xy' + 2y = 0, \qquad y(0) = 0, \quad y'(0) = 1; \qquad \text{see Problem 7}$$

Solution

In Problem 7 the general solution was found to be

$$\begin{aligned} y(x) &= a_0\left(1 - \frac{x^2}{1} + \frac{x^4}{3\cdot 1} - \frac{x^6}{5\cdot 3\cdot 1} + \cdots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{4\cdot 2} - \frac{x^7}{6\cdot 4\cdot 2} + \cdots\right) \\ &= a_0\sum_{n=0}^{\infty}(-1)^n\frac{2^n n!}{(2n)!}x^{2n} + a_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n n!}. \end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = a_0\left(-\frac{2x}{1} + \frac{4x^3}{3\cdot 1} - \frac{6x^5}{5\cdot 3\cdot 1} + \cdots\right) + a_1\left(1 - \frac{3x^2}{2} + \frac{5x^4}{4\cdot 2} - \frac{7x^6}{6\cdot 4\cdot 2} + \cdots\right)$$

Now apply the initial conditions, $y(0) = 0$ and $y'(0) = 1$, to determine a_0 and a_1.

$$\begin{aligned} y(0) &= a_0 = 0 \\ y'(0) &= a_1 = 1 \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= x - \frac{x^3}{2} + \frac{x^5}{4\cdot 2} - \frac{x^7}{6\cdot 4\cdot 2} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n n!}. \end{aligned}$$

Below is a plot of the various partial sums versus x.