Problem 3

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' - xy' - y = 0, \qquad x_0 = 1$$

Solution

$x = 1$ is not a zero of the coefficient of y'', so $x = 1$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \to \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \to \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Make it so that $x - 1$ appears in front of the second sum.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Bring $x - 1$ into the second summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n$ in the second sum, $k = n - 1$ in the third sum, and $k = n$ in the fourth sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} (x-1)^k - \sum_{k=1}^{\infty} k a_k (x-1)^k - \sum_{k+1=1}^{\infty} (k+1) a_{k+1} (x-1)^k - \sum_{k=0}^{\infty} a_k (x-1)^k = 0$$

Solve for k in the first and third sums, and start the second sum from $k = 0$.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k - \sum_{k=0}^{\infty} ka_k(x-1)^k - \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k - \sum_{k=0}^{\infty} a_k(x-1)^k = 0$$

Now that each sum has the same limits and $(x-1)^k$, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}(x-1)^k - ka_k(x-1)^k - (k+1)a_{k+1}(x-1)^k - a_k(x-1)^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - ka_k - (k+1)a_{k+1} - a_k](x-1)^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - (k+1)a_k - (k+1)a_{k+1}](x-1)^k = 0$$

$$\sum_{k=0}^{\infty}(k+1)[(k+2)a_{k+2} - a_k - a_{k+1}](x-1)^k = 0$$

The quantity in square brackets must be zero.

$$(k+2)a_{k+2} - a_k - a_{k+1} = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{a_{k+1} + a_k}{k+2}$$

Start plugging in values of k.

$$\begin{aligned}
k = 0: &\quad a_2 = \frac{1}{2}(a_1 + a_0) = \frac{1}{2}a_0 + \frac{1}{2}a_1 \\
k = 1: &\quad a_3 = \frac{1}{3}(a_2 + a_1) = \frac{1}{6}a_0 + \frac{1}{2}a_1 \\
k = 2: &\quad a_4 = \frac{1}{4}(a_3 + a_2) = \frac{1}{6}a_0 + \frac{1}{4}a_1 \\
k = 3: &\quad a_5 = \frac{1}{5}(a_4 + a_3) = \frac{1}{15}a_0 + \frac{3}{20}a_1 \\
&\quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= a_0 + a_1(x-1) + \left(\frac{1}{2}a_0 + \frac{1}{2}a_1\right)(x-1)^2 + \left(\frac{1}{6}a_0 + \frac{1}{2}a_1\right)(x-1)^3 + \left(\frac{1}{6}a_0 + \frac{1}{4}a_1\right)(x-1)^4 + \cdots \\
&= a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right] \\
&\qquad + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots\right] \\
&= a_0y_1(x) + a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right]\left[1 + (x-1) + \frac{3}{2}(x-1)^2 + (x-1)^3 + \cdots\right] \\
&\quad - \left[(x-1) + \frac{1}{2}(x-1)^2 + \frac{2}{3}(x-1)^3 + \cdots\right]\left[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots\right]
\end{aligned}$$

At $x = 1$ the Wronskian is nonzero,

$$W(y_1, y_2)(1) = (1 + 0 + 0 + \cdots)(1 + 0 + 0 + \cdots) - (0 + 0 + \cdots)(0 + 0 + \cdots) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.