

### Problem 4

In each of Problems 1 through 14:

- (a) Seek power series solutions of the given differential equation about the given point  $x_0$ ; find the recurrence relation.
- (b) Find the first four terms in each of two solutions  $y_1$  and  $y_2$  (unless the series terminates sooner).
- (c) By evaluating the Wronskian  $W(y_1, y_2)(x_0)$ , show that  $y_1$  and  $y_2$  form a fundamental set of solutions.
- (d) If possible, find the general term in each solution.

$$y'' + k^2x^2y = 0, \quad x_0 = 0, \quad k \text{ a constant}$$

#### Solution

$x_0 = 0$  is an ordinary point, so the solution can be represented as a power series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to  $x$  to get  $y'$  and  $y''$ .

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring  $k^2x^2$  into the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

Substitute  $p = n - 4$  in the first sum and  $p = n$  in the second sum to make the factors of  $x$  the same.

$$\sum_{p+4=2}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Solve for  $p$ .

$$\sum_{p=-2}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Write out the first two terms of the first sum.

$$(2)(1)a_2 + (3)(2)a_3x + \sum_{p=0}^{\infty} (p+4)(p+3) a_{p+4} x^{p+2} + \sum_{p=0}^{\infty} k^2 a_p x^{p+2} = 0$$

Now that the limits of each sum are the same and they have the same factors of  $x$ , the sums can be combined.

$$2a_2 + 6a_3x + \sum_{p=0}^{\infty} [(p+4)(p+3)a_{p+4}x^{p+2} + k^2a_px^{p+2}] = 0$$

Factor the summand.

$$2a_2 + 6a_3x + \sum_{p=0}^{\infty} [(p+4)(p+3)a_{p+4} + k^2a_p]x^{p+2} = 0 + 0x + 0x^2 + \dots$$

Match the coefficients on both sides of the equation.

$$\begin{aligned} 2a_2 &= 0 \\ 6a_3 &= 0 \\ (p+4)(p+3)a_{p+4} + k^2a_p &= 0 \end{aligned}$$

As a result,

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_{p+4} &= -\frac{k^2}{(p+4)(p+3)}a_p. \end{aligned}$$

Plug in enough values of  $p$  to get four terms involving  $a_0$  and four terms involving  $a_1$ .

$$\begin{array}{ll} p = 2 : & a_6 = -\frac{k^2}{6 \cdot 5}a_2 = 0 \\ p = 6 : & a_{10} = -\frac{k^2}{10 \cdot 9}a_6 = 0 \\ p = 10 : & a_{14} = -\frac{k^2}{14 \cdot 13}a_{10} = 0 \\ p = 14 : & a_{18} = -\frac{k^2}{18 \cdot 17}a_{14} = 0 \\ & \vdots \\ & a_{4m+2} = 0 \end{array} \qquad \begin{array}{ll} p = 3 : & a_7 = -\frac{k^2}{7 \cdot 6}a_3 = 0 \\ p = 7 : & a_{11} = -\frac{k^2}{11 \cdot 10}a_7 = 0 \\ p = 11 : & a_{15} = -\frac{k^2}{15 \cdot 14}a_{11} = 0 \\ p = 15 : & a_{19} = -\frac{k^2}{19 \cdot 18}a_{15} = 0 \\ & \vdots \\ & a_{4m+3} = 0 \end{array}$$

$$p = 0: a_4 = -\frac{k^2}{4 \cdot 3} a_0$$

$$p = 4: a_8 = -\frac{k^2}{8 \cdot 7} a_4 = \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3} a_0$$

$$p = 8: a_{12} = -\frac{k^2}{12 \cdot 11} a_8 = -\frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} a_0$$

$$p = 12: a_{16} = -\frac{k^2}{16 \cdot 15} a_{12} = \frac{k^8}{16 \cdot 15 \cdot 12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} a_0$$

$$\vdots$$

$$\begin{aligned} a_{4m} &= (-1)^m \frac{k^{2m}}{[(4m)(4m-4)\cdots(4)][(4m-1)(4m-5)\cdots(3)]} a_0 \\ &= (-1)^m \frac{k^{2m}}{[4^m(m)(m-1)\cdots(1)][4^m(m-\frac{1}{4})(m-\frac{5}{4})\cdots(\frac{3}{4})]} a_0 \\ &= (-1)^m \frac{k^{2m}}{[4^m m!][4^m \frac{\Gamma(m+\frac{3}{4})}{\Gamma(\frac{3}{4})}]} a_0 \\ &= (-1)^m \frac{k^{2m} \Gamma(\frac{3}{4})}{4^{2m} m! \Gamma(m+\frac{3}{4})} a_0 \\ &= \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma(\frac{3}{4})}{\Gamma(m+\frac{3}{4})} a_0 \end{aligned}$$

$$p = 1: a_5 = -\frac{k^2}{5 \cdot 4} a_1$$

$$p = 5: a_9 = -\frac{k^2}{9 \cdot 8} a_5 = \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4} a_1$$

$$p = 9: a_{13} = -\frac{k^2}{13 \cdot 12} a_9 = -\frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} a_1$$

$$p = 13: a_{17} = -\frac{k^2}{17 \cdot 16} a_{13} = \frac{k^8}{17 \cdot 16 \cdot 13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} a_1$$

$$\vdots$$

$$\begin{aligned} a_{4m+1} &= (-1)^m \frac{k^{2m}}{[(4m+1)(4m-3)\cdots(5)][(4m)(4m-4)\cdots(4)]} a_1 \\ &= (-1)^m \frac{k^{2m}}{[4^m(m+\frac{1}{4})(m-\frac{3}{4})\cdots(\frac{5}{4})][4^m(m)(m-1)\cdots(1)]} a_1 \\ &= (-1)^m \frac{k^{2m}}{\left[4^m \frac{\Gamma(m+\frac{5}{4})}{\Gamma(\frac{5}{4})}\right][4^m m!]} a_1 \\ &= (-1)^m \frac{k^{2m} \Gamma(\frac{5}{4})}{4^{2m} m! \Gamma(m+\frac{5}{4})} a_1 \\ &= \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma(\frac{5}{4})}{\Gamma(m+\frac{5}{4})} a_1 \end{aligned}$$

Therefore,

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 &= \sum_{m=0}^{\infty} a_{4m} x^{4m} + \sum_{m=0}^{\infty} a_{4m+1} x^{4m+1} + \sum_{m=0}^{\infty} a_{4m+2} x^{4m+2} + \sum_{m=0}^{\infty} a_{4m+3} x^{4m+3} \\
 &= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(m + \frac{3}{4}\right)} x^{4m} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{k}{4}\right)^{2m} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(m + \frac{5}{4}\right)} x^{4m+1} \\
 &= a_0 \left( 1 - \frac{k^2}{4 \cdot 3} x^4 + \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3} x^8 - \frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{12} + \frac{k^8}{16 \cdot 15 \cdot 12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{16} - \dots \right) \\
 &\quad + a_1 \left( x - \frac{k^2}{5 \cdot 4} x^5 + \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4} x^9 - \frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{13} + \frac{k^8}{17 \cdot 16 \cdot 13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{17} - \dots \right) \\
 &= a_0 y_1(x) + a_1 y_2(x).
 \end{aligned}$$

Now calculate the Wronskian of  $y_1$  and  $y_2$ .

$$\begin{aligned}
 W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
 &= y_1 y_2' - y_1' y_2 \\
 &= \left( 1 - \frac{k^2}{4 \cdot 3} x^4 + \frac{k^4}{8 \cdot 7 \cdot 4 \cdot 3} x^8 - \frac{k^6}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{12} + \frac{k^8}{16 \cdot 15 \cdot 12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{16} - \dots \right) \\
 &\quad \times \left( 1 - \frac{k^2}{4} x^4 + \frac{k^4}{8 \cdot 5 \cdot 4} x^8 - \frac{k^6}{12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{12} + \frac{k^8}{16 \cdot 13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{16} - \dots \right) \\
 &\quad - \left( -\frac{k^2}{3} x^3 + \frac{k^4}{7 \cdot 4 \cdot 3} x^7 - \frac{k^6}{11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{11} + \frac{k^8}{15 \cdot 12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} x^{15} - \dots \right) \\
 &\quad \times \left( x - \frac{k^2}{5 \cdot 4} x^5 + \frac{k^4}{9 \cdot 8 \cdot 5 \cdot 4} x^9 - \frac{k^6}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{13} + \frac{k^8}{17 \cdot 16 \cdot 13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} x^{17} - \dots \right)
 \end{aligned}$$

At  $x = 0$  the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that  $y_1$  and  $y_2$  form a fundamental set of solutions for the ODE.