Problem 5

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(1-x)y'' + y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Start the second sum from $n = 1$. This can be done because $n - 1$ is in the summand.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n - 1$ in the second sum, and $k = n$ in the third sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k+1=1}^{\infty} (k+1)k a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty}k(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty}a_kx^k = 0$$

Now that each sum has the same limits and factor of x, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k - k(k+1)a_{k+1}x^k + a_kx^k] = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + a_k]x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k}{k+2}a_{k+1} - \frac{a_k}{(k+2)(k+1)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: \quad & a_2 = -\frac{a_0}{2\cdot 1} \\
k=1: \quad & a_3 = \frac{1}{3}a_2 - \frac{a_1}{3\cdot 2} = -\frac{a_0}{3\cdot 2\cdot 1} - \frac{a_1}{3\cdot 2} \\
k=2: \quad & a_4 = \frac{2}{4}a_3 - \frac{a_2}{4\cdot 3} = -\frac{2a_0}{4\cdot 3\cdot 2\cdot 1} - \frac{2a_1}{4\cdot 3\cdot 2} + \frac{a_0}{4\cdot 3\cdot 2\cdot 1} = -\frac{a_0}{4\cdot 3\cdot 2\cdot 1} - \frac{2a_1}{4\cdot 3\cdot 2} \\
k=3: \quad & a_5 = \frac{3}{5}a_4 - \frac{a_3}{5\cdot 4} = -\frac{3a_0}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{3\cdot 2a_1}{5\cdot 4\cdot 3\cdot 2} + \frac{a_0}{5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{a_1}{5\cdot 4\cdot 3\cdot 2} \\
& \phantom{a_5 = \frac{3}{5}a_4 - \frac{a_3}{5\cdot 4}} = -\frac{2a_0}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{5a_1}{5\cdot 4\cdot 3\cdot 2} \\
k=4: \quad & a_6 = \frac{4}{6}a_5 - \frac{a_4}{6\cdot 5} = -\frac{4\cdot 2a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{5\cdot 4a_1}{6\cdot 5\cdot 4\cdot 3\cdot 2} + \frac{a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{2a_1}{6\cdot 5\cdot 4\cdot 3\cdot 2} \\
& \phantom{a_6 = \frac{4}{6}a_5 - \frac{a_4}{6\cdot 5}} = -\frac{7a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{18a_1}{6\cdot 5\cdot 4\cdot 3\cdot 2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}a_nx^n \\
&= a_0 + a_1x - \frac{a_0}{2\cdot 1}x^2 + \left(-\frac{a_0}{3\cdot 2\cdot 1} - \frac{a_1}{3\cdot 2}\right)x^3 + \left(-\frac{a_0}{4\cdot 3\cdot 2\cdot 1} - \frac{2a_1}{4\cdot 3\cdot 2}\right)x^4 \\
&\quad + \left(-\frac{2a_0}{5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{5a_1}{5\cdot 4\cdot 3\cdot 2}\right)x^5 + \left(-\frac{7a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{18a_1}{6\cdot 5\cdot 4\cdot 3\cdot 2}\right)x^6 + \cdots \\
&= a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \cdots\right) + a_1\left(x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{24} - \cdots\right) \\
&= a_0y_1(x) + a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left(1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \cdots\right)\left(1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{5x^4}{24} - \cdots\right) \\ &\quad - \left(-x - \frac{x^2}{2} - \frac{x^3}{6} - \cdots\right)\left(x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{24} - \cdots\right) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1 - 0 - 0 - \cdots)(1 - 0 - 0 - \cdots) - 0 = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.