Problem 6

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(2+x^2)y'' - xy' + 4y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 2, x^2, x, and 4 into the respective summands.

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the other sums.

$$\sum_{k+2=2}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$$

Solve for k in the first sum. The second and third sums can be set to start from $k = 0$ because of the k and $k - 1$ factors.

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$$

Since each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2}x^k + k(k-1)a_kx^k - ka_kx^k + 4a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + k(k-1)a_k - ka_k + 4a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)a_{k+2} + (k^2-2k+4)a_k]x^k = 0$$

The coefficients must be zero.

$$2(k+2)(k+1)a_{k+2} + (k^2-2k+4)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = -\frac{k^2-2k+4}{2(k+2)(k+1)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = -a_0 \\
k=1: &\quad a_3 = -\frac{1}{4}a_1 \\
k=2: &\quad a_4 = -\frac{1}{6}a_2 = \frac{1}{6}a_0 \\
k=3: &\quad a_5 = -\frac{7}{40}a_3 = \frac{7}{160}a_1 \\
k=4: &\quad a_6 = -\frac{1}{5}a_4 = -\frac{1}{30}a_0 \\
k=5: &\quad a_7 = -\frac{19}{84}a_5 = -\frac{19}{1920}a_1 \\
k=6: &\quad a_8 = -\frac{1}{4}a_6 = \frac{1}{120}a_0 \\
k=7: &\quad a_9 = -\frac{13}{48}a_7 = \frac{247}{92\,160}a_1 \\
&\quad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}a_nx^n \\
&= a_0 + a_1x - a_0x^2 - \frac{1}{4}a_1x^3 + \frac{1}{6}a_0x^4 + \frac{7}{160}a_1x^5 - \frac{1}{30}a_0x^6 - \frac{19}{1920}a_1x^7 + \frac{1}{120}a_0x^8 + \frac{247}{92\,160}a_1x^9 + \cdots \\
&= a_0\left(1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots\right) + a_1\left(x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots\right) \\
&= a_0y_1(x) + a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left(1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots\right)\left(1 - \frac{3x^2}{4} + \frac{7x^4}{32} - \frac{133x^6}{1920} + \cdots\right) \\ &\qquad - \left(-2x + \frac{2x^3}{3} - \frac{x^5}{5} + \cdots\right)\left(x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots\right) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1 - 0 + 0 - \cdots)(1 - 0 + 0 - \cdots) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.