# Problem 9

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point $x_0$; find the recurrence relation.

(b) Find the first four terms in each of two solutions $y_1$ and $y_2$ (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that $y_1$ and $y_2$ form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(1+x^2)y'' - 4xy' + 6y = 0, \qquad x_0 = 0$$

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**Solution**

$x = 0$ is not a zero of the coefficient of $y''$, so $x = 0$ is an ordinary point. As such, the solution for $y$ can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to $x$ to get $y'$ and $y''$.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring $x^2$, $4x$, and 6 into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n} - \sum_{n=1}^{\infty} 4n a_n x^{n} + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

Start the second and third sums from $n = 0$. This can be done because of the $n$ and $n-1$ factors.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^{n} - \sum_{n=0}^{\infty} 4n a_n x^{n} + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

Substitute $k = n-2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2} x^{k} + \sum_{k=0}^{\infty} k(k-1)a_k x^{k} - \sum_{k=0}^{\infty} 4k a_k x^{k} + \sum_{k=0}^{\infty} 6a_k x^k = 0$$

Solve for $k$.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k+\sum_{k=0}^{\infty}k(k-1)a_kx^k-\sum_{k=0}^{\infty}4ka_kx^k+\sum_{k=0}^{\infty}6a_kx^k=0$$

Now that each of the sums has the same limits and factors of $x$, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k+k(k-1)a_kx^k-4ka_kx^k+6a_kx^k]=0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}+k(k-1)a_k-4ka_k+6a_k]x^k=0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}+(k^2-5k+6)a_k]x^k=0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}+(k-3)(k-2)a_k]x^k=0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2}+(k-3)(k-2)a_k=0$$

Solve for $a_{k+2}$.

$$a_{k+2}=-\frac{(k-3)(k-2)}{(k+2)(k+1)}a_k$$

Plug in enough values of $k$ to get four terms involving $a_0$ and four terms involving $a_1$.

$$\begin{aligned}
k=0:&\quad a_2=-\frac{(-3)(-2)}{(2)(1)}a_0=-3a_0\\
k=1:&\quad a_3=-\frac{(-2)(-1)}{(3)(2)}a_1=-\frac{1}{3}a_1\\
k=2:&\quad a_4=-\frac{(-1)(0)}{(4)(3)}a_2=0\\
k=3:&\quad a_5=-\frac{(0)(1)}{(5)(4)}a_3=0\\
k=4:&\quad a_6=-\frac{(1)(2)}{(6)(5)}a_4=0\\
&\quad\vdots
\end{aligned}$$

Note that $a_k=0$ for $k>3$. Therefore,

$$\begin{aligned}
y(x)&=\sum_{n=0}^{\infty}a_nx^n\\
&=a_0+a_1x-3a_0x^2-\frac{1}{3}a_1x^3\\
&=a_0(1-3x^2)+a_1\left(x-\frac{x^3}{3}\right)\\
&=a_0y_1(x)+a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of $y_1$ and $y_2$.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= (1 - 3x^2)(1 - x^2) - (-6x)\left(x - \frac{x^3}{3}\right) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that $y_1$ and $y_2$ form a fundamental set of solutions for the ODE.