Problem 7

In each of Problems 5 through 8, determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

$$(1+x^3)y'' + 4xy' + y = 0; \qquad x_0 = 0, \quad x_0 = 2$$

Solution

The coefficient of y'' is $1 + x^3$. Determine where its zeros are located.

$$1 + x^3 = 0$$

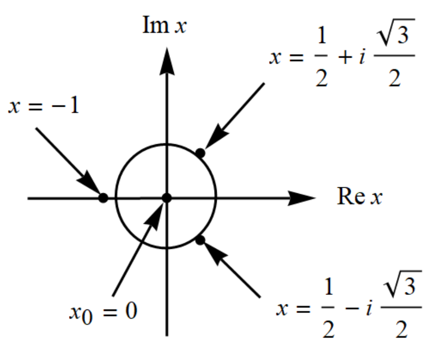
$$x^3 = -1$$

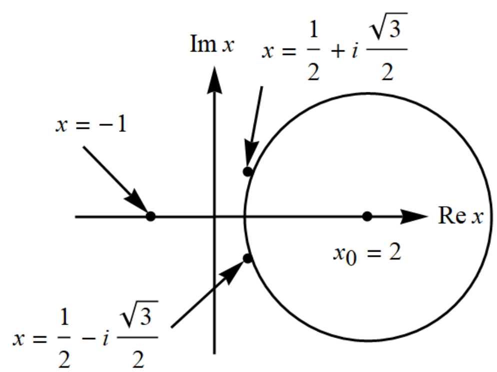
$$\begin{aligned} x &= (-1)^{1/3} \\ &= (e^{i\pi + 2in\pi})^{1/3}, \quad n = 0, \pm 1, \pm 2, \ldots \\ &= e^{i\pi/3 + 2in\pi/3} \end{aligned}$$

Distinct zeros occur if $n = 0$, $n = 1$, and $n = 2$. Other values of n lead to redundant values.

$$\begin{aligned} n = 0: &\quad x = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ n = 1: &\quad x = e^{i\pi} = \cos\pi + i\sin\pi = -1 \\ n = 2: &\quad x = e^{5i\pi/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

Plot these zeros in the complex plane and expand a circle centered at x_0 as much as possible until it intersects one of them.





To determine the radii convergence, use the distance formula for the points, $(0,0)$ and $(1/2,\sqrt{3}/2)$, in the first case and the points, $(2,0)$ and $(1/2,\sqrt{3}/2)$, in the second case.

$$x_0 = 0: \quad d = \sqrt{\left(\frac{1}{2}-0\right)^2 + \left(\frac{\sqrt{3}}{2}-0\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$x_0 = 2: \quad d = \sqrt{\left(\frac{1}{2}-2\right)^2 + \left(\frac{\sqrt{3}}{2}-0\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$$

If $x_0 = 0$, the lower bound for the radius of convergence is 1. If $x_0 = 2$, the lower bound for the radius of convergence is $\sqrt{3}$.