Problem 11

For each of the differential equations in Problems 11 through 14, find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

$$y'' + (\sin x)y = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

The coefficient of y'' has no zeros, in fact, so this solution is expected to be valid for $-\infty < x < \infty$; that is, the radius of convergence is ∞. Differentiate this series with respect to x twice to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these expressions into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + (\sin x) \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute the Taylor series expansion of $\sin x$ about $x = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right] \sum_{n=0}^{\infty} a_n x^n = 0$$

Multiply the two series together.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} a_{n-k} x^{n-k} = 0$$

Substitute $p = n - 2$ in the first sum and $p = n$ in the second sum.

$$\sum_{p+2=2}^{\infty} (p+2)(p+1) a_{p+2} x^p + \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{(-1)^k}{(2k+1)!} x^{2k+1} a_{p-k} x^{p-k} = 0$$

$$\sum_{p=0}^{\infty} (p+2)(p+1) a_{p+2} x^p + \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{(-1)^k}{(2k+1)!} a_{p-k} x^{k+1} x^p = 0$$

Combine the two sums now that they have the same limits.

$$\sum_{p=0}^{\infty} \left[(p+2)(p+1) a_{p+2} x^p + \sum_{k=0}^{p} \frac{(-1)^k}{(2k+1)!} a_{p-k} x^{k+1} x^p\right] = 0$$

Factor x^p.

$$\sum_{p=0}^{\infty}\left[(p+2)(p+1)a_{p+2}+\sum_{k=0}^{p}\frac{(-1)^k}{(2k+1)!}a_{p-k}x^{k+1}\right]x^p=0$$

Write out the first few terms of the sum.

$$\begin{aligned}\left[(2)(1)a_2+\sum_{k=0}^{0}\frac{(-1)^k}{(2k+1)!}a_{-k}x^{k+1}\right] &+\left[(3)(2)a_3+\sum_{k=0}^{1}\frac{(-1)^k}{(2k+1)!}a_{1-k}x^{k+1}\right]x\\ &+\left[(4)(3)a_4+\sum_{k=0}^{2}\frac{(-1)^k}{(2k+1)!}a_{2-k}x^{k+1}\right]x^2\\ &+\left[(5)(4)a_5+\sum_{k=0}^{3}\frac{(-1)^k}{(2k+1)!}a_{3-k}x^{k+1}\right]x^3\\ &+\left[(6)(5)a_6+\sum_{k=0}^{4}\frac{(-1)^k}{(2k+1)!}a_{4-k}x^{k+1}\right]x^4\\ &+\left[(7)(6)a_7+\sum_{k=0}^{5}\frac{(-1)^k}{(2k+1)!}a_{5-k}x^{k+1}\right]x^5\\ &+\left[(8)(7)a_8+\sum_{k=0}^{6}\frac{(-1)^k}{(2k+1)!}a_{6-k}x^{k+1}\right]x^6+\cdots=0\end{aligned}$$

Expand each term.

$$\begin{aligned}&(2a_2+a_0x)+\left(6a_3+a_1x-\frac{1}{3!}a_0x^2\right)x+\left(12a_4+a_2x-\frac{1}{3!}a_1x^2+\frac{1}{5!}a_0x^3\right)x^2\\ &+\left(20a_5+a_3x-\frac{1}{3!}a_2x^2+\frac{1}{5!}a_1x^3-\frac{1}{7!}a_0x^4\right)x^3+\left(30a_6+a_4x-\frac{1}{3!}a_3x^2+\frac{1}{5!}a_2x^3-\frac{1}{7!}a_1x^4+\frac{1}{9!}a_0x^5\right)x^4\\ &+\left(42a_7+a_5x-\frac{1}{3!}a_4x^2+\frac{1}{5!}a_3x^3-\frac{1}{7!}a_2x^4+\frac{1}{9!}a_1x^5-\frac{1}{11!}a_0x^6\right)x^5\\ &+\left(56a_8+a_6x-\frac{1}{3!}a_5x^2+\frac{1}{5!}a_4x^3-\frac{1}{7!}a_3x^4+\frac{1}{9!}a_2x^5-\frac{1}{11!}a_1x^6+\frac{1}{13!}a_0x^7\right)x^6+\cdots=0\end{aligned}$$

Write both sides in powers of x.

$$\begin{aligned}2a_2+(a_0+6a_3)x+(a_1+12a_4)x^2+\left(-\frac{1}{3!}a_0+a_2+20a_5\right)x^3&\\ +\left(-\frac{1}{3!}a_1+a_3+30a_6\right)x^4+\left(\frac{1}{5!}a_0-\frac{1}{3!}a_2+a_4+42a_7\right)x^5&\\ +\left(\frac{1}{5!}a_1-\frac{1}{3!}a_3+a_5+56a_8\right)x^6+\cdots=0+0x+0x^2+\cdots&\end{aligned}$$

Match the coefficients on both sides.

$$\begin{aligned}
2a_2 &= 0 \\
a_0 + 6a_3 &= 0 \\
a_1 + 12a_4 &= 0 \\
-\frac{1}{3!}a_0 + a_2 + 20a_5 &= 0 \\
-\frac{1}{3!}a_1 + a_3 + 30a_6 &= 0 \\
\frac{1}{5!}a_0 - \frac{1}{3!}a_2 + a_4 + 42a_7 &= 0 \\
\frac{1}{5!}a_1 - \frac{1}{3!}a_3 + a_5 + 56a_8 &= 0 \\
&\vdots
\end{aligned}$$

Solving this system of equations yields

$$\begin{aligned}
a_2 &= 0 \\
a_3 &= -\frac{1}{6}a_0 \\
a_4 &= -\frac{1}{12}a_1 \\
a_5 &= \frac{1}{120}a_0 \\
a_6 &= \frac{1}{180}a_0 + \frac{1}{180}a_1 \\
a_7 &= -\frac{1}{5040}a_0 + \frac{1}{504}a_1 \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x - \frac{1}{6}a_0 x^3 - \frac{1}{12}a_1 x^4 + \frac{1}{120}a_0 x^5 + \left(\frac{1}{180}a_0 + \frac{1}{180}a_1\right)x^6 + \left(-\frac{1}{5040}a_0 + \frac{1}{504}a_1\right)x^7 + \cdots \\
&= a_0\left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \cdots\right) + a_1\left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \cdots\right)\left(1 - \frac{1}{12}\cdot 4x^3 + \frac{1}{180}\cdot 6x^5 + \frac{1}{504}\cdot 7x^6 + \cdots\right) \\ &\quad - \left(-\frac{1}{6}\cdot 3x^2 + \frac{1}{120}\cdot 5x^4 + \frac{1}{180}\cdot 6x^5 - \cdots\right)\left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots\right) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.