Problem 13

For each of the differential equations in Problems 11 through 14, find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

$$(\cos x)y'' + xy' - 2y = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

The coefficient of y'' has zeros at $-\pi/2$ and $\pi/2$, in fact, so this solution is expected to be valid for $-\pi/2 < x < \pi/2$; that is, the radius of convergence is at least $\pi/2$. Differentiate this series with respect to x twice to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these expressions into the ODE.

$$(\cos x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x and 2 into the respective summands.

$$(\cos x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Because of n, the second sum can be set to start from $n = 0$.

$$(\cos x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Substitute $p = n - 2$ in the first sum and $p = n$ in the others.

$$(\cos x)\sum_{p+2=2}^{\infty} (p+2)(p+1)a_{p+2} x^{p} + \sum_{p=0}^{\infty} p a_p x^{p} - \sum_{p=0}^{\infty} 2a_p x^p = 0$$

Solve for p.

$$(\cos x)\sum_{p=0}^{\infty} (p+2)(p+1)a_{p+2} x^{p} + \sum_{p=0}^{\infty} p a_p x^{p} - \sum_{p=0}^{\infty} 2a_p x^p = 0$$

Substitute the Taylor series expansion of $\cos x$ about $x = 0$.

$$\left[\sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} x^{2p}\right]\sum_{p=0}^{\infty} (p+2)(p+1)a_{p+2} x^{p} + \sum_{p=0}^{\infty} p a_p x^{p} - \sum_{p=0}^{\infty} 2a_p x^p = 0$$

Multiply the two series together.

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{(-1)^{p-k}}{[2(p-k)]!}x^{2(p-k)}(k+2)(k+1)a_{k+2}x^k+\sum_{p=0}^{\infty}pa_px^p-\sum_{p=0}^{\infty}2a_px^p=0$$

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{(-1)^{p-k}}{[2(p-k)]!}x^{p-k}(k+2)(k+1)a_{k+2}x^p+\sum_{p=0}^{\infty}pa_px^p-\sum_{p=0}^{\infty}2a_px^p=0$$

Now that the limits are the same, combine the sums.

$$\sum_{p=0}^{\infty}\left\{\sum_{k=0}^{p}\frac{(-1)^{p-k}}{[2(p-k)]!}x^{p-k}(k+2)(k+1)a_{k+2}x^p+pa_px^p-2a_px^p\right\}=0$$

Factor x^p.

$$\sum_{p=0}^{\infty}\left\{\sum_{k=0}^{p}\frac{(-1)^{p-k}}{[2(p-k)]!}x^{p-k}(k+2)(k+1)a_{k+2}+(p-2)a_p\right\}x^p=0$$

Write out the first few terms of the sum.

$$\begin{aligned}&\left\{\sum_{k=0}^{0}\frac{(-1)^{-k}}{[2(-k)]!}x^{-k}(k+2)(k+1)a_{k+2}+(-2)a_0\right\}\\&\quad+\left\{\sum_{k=0}^{1}\frac{(-1)^{1-k}}{[2(1-k)]!}x^{1-k}(k+2)(k+1)a_{k+2}+(-1)a_1\right\}x\\&\quad+\left\{\sum_{k=0}^{2}\frac{(-1)^{2-k}}{[2(2-k)]!}x^{2-k}(k+2)(k+1)a_{k+2}+(0)a_2\right\}x^2\\&\quad+\left\{\sum_{k=0}^{3}\frac{(-1)^{3-k}}{[2(3-k)]!}x^{3-k}(k+2)(k+1)a_{k+2}+(1)a_3\right\}x^3\\&\quad+\left\{\sum_{k=0}^{4}\frac{(-1)^{4-k}}{[2(4-k)]!}x^{4-k}(k+2)(k+1)a_{k+2}+(2)a_4\right\}x^4\\&\quad+\left\{\sum_{k=0}^{5}\frac{(-1)^{5-k}}{[2(5-k)]!}x^{5-k}(k+2)(k+1)a_{k+2}+(3)a_5\right\}x^5\\&\quad+\left\{\sum_{k=0}^{6}\frac{(-1)^{6-k}}{[2(6-k)]!}x^{6-k}(k+2)(k+1)a_{k+2}+(4)a_6\right\}x^6+\cdots=0\end{aligned}$$

Expand each term.

$$\begin{aligned}&(2a_2-2a_0)+(-a_2x+6a_3-a_1)x+\left(\frac{1}{12}a_2x^2-3a_3x+12a_4\right)x^2\\&\quad+\left(-\frac{1}{360}a_2x^3+\frac{1}{4}a_3x^2-6a_4x+20a_5+a_3\right)x^3\\&\quad+\left(\frac{1}{20\,160}a_2x^4-\frac{1}{120}a_3x^3+\frac{1}{2}a_4x^2-10a_5x+30a_6+2a_4\right)x^4\\&\quad+\left(-\frac{1}{1\,814\,400}a_2x^5+\frac{1}{6720}a_3x^4-\frac{1}{60}a_4x^3+\frac{5}{6}a_5x^2-15a_6x+42a_7+3a_5\right)x^5+\cdots=0\end{aligned}$$

Write both sides in powers of x.

$$\begin{aligned}(2a_2 - 2a_0) + (6a_3 - a_1)x + (12a_4 - a_2)x^2 + (-3a_3 + 20a_5 + a_3)x^3 \\ + \left(\frac{1}{12}a_2 - 6a_4 + 30a_6 + 2a_4\right)x^4 \\ + \left(\frac{1}{4}a_3 - 10a_5 + 42a_7 + 3a_5\right)x^5 + \cdots = 0 + 0x + 0x^2 + \cdots\end{aligned}$$

Match the coefficients on both sides.

$$\begin{aligned}2a_2 - 2a_0 &= 0 \\ 6a_3 - a_1 &= 0 \\ 12a_4 - a_2 &= 0 \\ -3a_3 + 20a_5 + a_3 &= 0 \\ \frac{1}{12}a_2 - 6a_4 + 30a_6 + 2a_4 &= 0 \\ \frac{1}{4}a_3 - 10a_5 + 42a_7 + 3a_5 &= 0 \\ &\vdots\end{aligned}$$

Solving this system of equations yields

$$\begin{aligned}a_2 &= a_0 \\ a_3 &= \frac{1}{6}a_1 \\ a_4 &= \frac{1}{12}a_2 = \frac{1}{12}a_0 \\ a_5 &= \frac{1}{10}a_3 = \frac{1}{60}a_1 \\ a_6 &= \frac{1}{120}a_0 \\ a_7 &= \frac{1}{560}a_1 \\ &\vdots\end{aligned}$$

Therefore,

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_0 x^2 + \frac{1}{6}a_1 x^3 + \frac{1}{12}a_0 x^4 + \frac{1}{60}a_1 x^5 + \frac{1}{120}a_0 x^6 + \frac{1}{560}a_1 x^7 + \cdots \\ &= a_0\left(1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \cdots\right) \\ &= a_0 y_1(x) + a_1 y_2(x).\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= y_1 y_2' - y_1' y_2 \\ &= \left(1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots\right)\left(1 + \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{80}x^6 + \cdots\right) \\ &\quad - \left(2x + \frac{1}{3}x^3 + \frac{1}{20}x^5 + \cdots\right)\left(x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \cdots\right) \end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.