Problem 15

Let x and x^2 be solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point? Prove your answer.

Solution

If x and x^2 are solutions of the ODE, then they satisfy

$$\begin{cases} P(x)(x)'' + Q(x)(x)' + R(x)(x) = 0 \\ P(x)(x^2)'' + Q(x)(x^2)' + R(x)(x^2) = 0 \end{cases}$$

$$\begin{cases} P(x)(0) + Q(x)(1) + R(x)(x) = 0 \\ P(x)(2) + Q(x)(2x) + R(x)(x^2) = 0 \end{cases}$$

$$\begin{cases} Q(x) + xR(x) = 0 \\ 2P(x) + 2xQ(x) + x^2R(x) = 0 \end{cases}.$$

Solve for $Q(x)$ in the first equation

$$Q(x) = -xR(x)$$

and plug it into the second equation.

$$2P(x) + 2x[-xR(x)] + x^2R(x) = 0$$

Solve for $P(x)$.

$$P(x) = \frac{1}{2}x^2R(x)$$

Substitute this formula for $P(x)$ into the original ODE.

$$\left[\frac{1}{2}x^2R(x)\right]y'' + Q(x)y' + R(x)y = 0$$

Because $x = 0$ is a zero of the coefficient of y'', it is a singular point.