Problem 18

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 16 through 21, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

$$y' = e^{x^2} y, \qquad \text{three terms only}$$

Solution

The coefficient of y' has no zeros, so $x = 0$ is an ordinary point. As such, the solution can be represented as a power series about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate it with respect to x to get y'.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plug these expressions into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = e^{x^2} \sum_{n=0}^{\infty} a_n x^n$$

Substitute $p = n - 1$ in the sum on the left and $p = n$ in the sum on the right.

$$\sum_{p+1=1}^{\infty} (p+1) a_{p+1} x^p = e^{x^2} \sum_{p=0}^{\infty} a_p x^p$$

Solve for p and bring all terms to the left side.

$$\sum_{p=0}^{\infty} (p+1) a_{p+1} x^p - e^{x^2} \sum_{p=0}^{\infty} a_p x^p = 0$$

Substitute the Taylor series expansion of e^{x^2} about $x = 0$.

$$\sum_{p=0}^{\infty} (p+1) a_{p+1} x^p - \left(\sum_{p=0}^{\infty} \frac{x^{2p}}{p!} \right) \sum_{p=0}^{\infty} a_p x^p = 0$$

Multiply the series together.

$$\sum_{p=0}^{\infty}(p+1)a_{p+1}x^p - \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{x^{2k}}{k!}a_{p-k}x^{p-k} = 0$$

Isolate x^p in the second sum.

$$\sum_{p=0}^{\infty}(p+1)a_{p+1}x^p - \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{x^{k}}{k!}a_{p-k}x^{p} = 0$$

Since the limits are the same, the sums can be combined.

$$\sum_{p=0}^{\infty}\left[(p+1)a_{p+1}x^p - \sum_{k=0}^{p}\frac{x^{k}}{k!}a_{p-k}x^{p}\right] = 0$$

Factor x^p.

$$\sum_{p=0}^{\infty}\left[(p+1)a_{p+1} - \sum_{k=0}^{p}\frac{x^{k}}{k!}a_{p-k}\right]x^p = 0$$

Write out the first few terms of this sum.

$$\begin{aligned}\left[(1)a_1 - \sum_{k=0}^{0}\frac{x^k}{k!}a_{-k}\right] + \left[(2)a_2 - \sum_{k=0}^{1}\frac{x^k}{k!}a_{1-k}\right]x \\ + \left[(3)a_3 - \sum_{k=0}^{2}\frac{x^k}{k!}a_{2-k}\right]x^2 + \left[(4)a_4 - \sum_{k=0}^{3}\frac{x^k}{k!}a_{3-k}\right]x^3 \\ + \left[(5)a_5 - \sum_{k=0}^{4}\frac{x^k}{k!}a_{4-k}\right]x^4 + \cdots = 0\end{aligned}$$

Expand each of these terms.

$$\begin{aligned}(a_1 - a_0) + (2a_2 - a_1 - a_0x)x + \left(3a_3 - a_2 - a_1x - \frac{1}{2}a_0x^2\right)x^2 \\ + \left(4a_4 - a_3 - a_2x - \frac{1}{2}a_1x^2 - \frac{1}{6}a_0x^3\right)x^3 + \left(5a_5 - a_4 - a_3x - \frac{1}{2}a_2x^2 - \frac{1}{6}a_1x^3 - \frac{1}{24}a_0x^4\right)x^4 + \cdots = 0\end{aligned}$$

Write both sides in powers of x.

$$\begin{aligned}(a_1 - a_0) + (2a_2 - a_1)x + (-a_0 + 3a_3 - a_2)x^2 + (-a_1 + 4a_4 - a_3)x^3 \\ + \left(-\frac{1}{2}a_0 - a_2 + 5a_5 - a_4\right)x^4 + \cdots = 0 + 0x + 0x^2 + \cdots\end{aligned}$$

Match the coefficients on both sides.

$$\begin{aligned}a_1 - a_0 &= 0 \\ 2a_2 - a_1 &= 0 \\ -a_0 + 3a_3 - a_2 &= 0 \\ -a_1 + 4a_4 - a_3 &= 0 \\ -\frac{1}{2}a_0 - a_2 + 5a_5 - a_4 &= 0 \\ &\vdots\end{aligned}$$

Solving this system of equations yields

$$\begin{aligned} a_1 &= a_0 \\ a_2 &= \frac{1}{2}a_0 \\ a_3 &= \frac{1}{2}a_0 \\ a_4 &= \frac{3}{8}a_0 \\ a_5 &= \frac{11}{40}a_0 \\ &\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_0 x + \frac{1}{2}a_0 x^2 + \frac{1}{2}a_0 x^3 + \frac{3}{8}a_0 x^4 + \frac{11}{40}a_0 x^5 + \cdots \\ &= a_0\left(1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \frac{11}{40}x^5 + \cdots\right). \end{aligned}$$