Problem 20

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 16 through 21, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

$$y' - y = x^2$$

Solution

The coefficient of y' has no zeros, so $x = 0$ is an ordinary point. As such, the solution can be represented as a power series about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate it with respect to x to get y'.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plug these expressions into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = x^2$$

Substitute $k = n - 1$ into the first sum and $k = n$ into the second sum.

$$\sum_{k+1=1}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} a_k x^k = x^2$$

Solve for k.

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} a_k x^k = x^2$$

Now that the limits are the same in both sums, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+1) a_{k+1} x^k - a_k x^k\right] = x^2$$

Factor x^k.

$$\sum_{k=0}^{\infty} \left[(k+1) a_{k+1} - a_k\right] x^k = x^2$$

The coefficients must be zero if $k \neq 2$ and must be 1 if $k = 2$.

$$(k+1)a_{k+1} - a_k = \begin{cases} 0 & \text{if } k \neq 2 \\ 1 & \text{if } k = 2 \end{cases}$$

That is,

$$\begin{aligned} (3)a_3 - a_2 &= 1 \\ (k+1)a_{k+1} - a_k &= 0. \end{aligned}$$

Solve for a_3 and a_{k+1}.

$$\begin{aligned} a_3 &= \frac{a_2}{3} + \frac{1}{3} \\ a_{k+1} &= \frac{a_k}{k+1} \end{aligned}$$

Plug in enough values for k to see a pattern and determine a_k.

$$\begin{aligned} k = 0: &\quad a_1 = \frac{a_0}{1} \\ k = 1: &\quad a_2 = \frac{a_1}{2} = \frac{a_0}{2 \cdot 1} \\ k = 2: &\quad a_3 = \frac{a_2}{3} + \frac{1}{3} = \frac{a_0}{3 \cdot 2 \cdot 1} + \frac{1}{3} \\ k = 3: &\quad a_4 = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3} \\ k = 4: &\quad a_5 = \frac{a_4}{5} = \frac{a_0}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3} \\ k = 5: &\quad a_6 = \frac{a_5}{6} = \frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{6 \cdot 5 \cdot 4 \cdot 3} \\ &\quad \vdots \\ &\quad a_k = \begin{cases} \dfrac{a_0}{k!} & \text{if } k \leq 2 \\ \dfrac{a_0}{k!} + \dfrac{2 \cdot 1}{k!} = \dfrac{a_0 + 2}{k!} & \text{if } k > 2 \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{2} \frac{a_0}{n!}x^n + \sum_{n=3}^{\infty} \frac{a_0+2}{n!}x^n \\
&= \sum_{n=0}^{2} \frac{a_0}{n!}x^n + \sum_{n=3}^{\infty} \left(\frac{a_0}{n!}x^n + \frac{2}{n!}x^n\right) \\
&= \sum_{n=0}^{2} \frac{a_0}{n!}x^n + \sum_{n=3}^{\infty} \frac{a_0}{n!}x^n + \sum_{n=3}^{\infty} \frac{2}{n!}x^n \\
&= \sum_{n=0}^{\infty} \frac{a_0}{n!}x^n + \sum_{n=3}^{\infty} \frac{2}{n!}x^n \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + 2\sum_{n=0}^{\infty} \frac{x^n}{n!} - x^2 - 2x - 2 \\
&= a_0 e^x + 2e^x - x^2 - 2x - 2 \\
&= (a_0+2)e^x - x^2 - 2x - 2 \\
&= A_0 e^x - x^2 - 2x - 2.
\end{aligned}$$

Now solve the ODE using a method from Chapter 2.

$$y' - y = x^2$$

Use an integrating factor I to solve it.

$$I = \exp\left[\int^x (-1)\,ds\right] = e^{-x}$$

Multiply both sides by I.

$$e^{-x}y' - e^{-x}y = x^2e^{-x}$$

The left side can be written as $d/dx(Iy)$.

$$\frac{d}{dx}(e^{-x}y) = x^2e^{-x}$$

Integrate both sides with respect to x, using integration by parts twice.

$$\begin{aligned}
e^{-x}y &= \int^x s^2 e^{-s}\,ds + C \\
&= \int^x s^2 \frac{d}{ds}(-e^{-s})\,ds + C \\
&= \left[s^2(-e^{-s})\bigg|^x - \int^x (2s)(-e^{-s})\,ds \right] + C \\
&= x^2(-e^{-x}) + \int^x (2s)e^{-s}\,ds + C \\
&= -x^2e^{-x} + \int^x (2s)\frac{d}{ds}(-e^{-s})\,ds + C \\
&= -x^2e^{-x} + \left[(2s)(-e^{-s})\bigg|^x - \int^x (2)(-e^{-s})\,ds \right] + C \\
&= -x^2e^{-x} + (2x)(-e^{-x}) + 2\int^x e^{-s}\,ds + C \\
&= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C
\end{aligned}$$

Therefore, multiplying both sides by e^x,

$$y(x) = -x^2 - 2x - 2 + Ce^x.$$