Problem 21

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 16 through 21, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

$$y' + xy = 1 + x$$

Solution

Series Solution Method

The coefficient of y' has no zeros, so $x = 0$ is an ordinary point. As such, the solution can be represented as a power series about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate it with respect to x to get y'.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \to \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plug these expressions into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 1 + x$$

Bring x into the summand.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 1 + x$$

Substitute $k + 1 = n - 1$ into the first sum and $k = n$ into the second sum.

$$\sum_{k+2=1}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 1 + x$$

Solve for k.

$$\sum_{k=-1}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 1 + x$$

Write out the first term of the first sum.

$$a_1 + \sum_{k=0}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} = 1 + x$$

Now that the limits are the same in both sums, they can be combined.

$$a_1 + \sum_{k=0}^{\infty} \left[(k+2)a_{k+2}x^{k+1} + a_k x^{k+1}\right] = 1 + x$$

Factor x^{k+1} and write the right side in powers of x.

$$a_1 + \sum_{k=0}^{\infty} \left[(k+2)a_{k+2} + a_k\right] x^{k+1} = 1 + x + 0x^2 + \cdots$$

Match the coefficients on both sides.

$$\begin{aligned} a_1 &= 1 \\ (2)a_2 + a_0 &= 1 \\ (k+2)a_{k+2} + a_k &= 0 \end{aligned}$$

Solve for a_{k+2}.

$$\begin{aligned} a_2 &= -\frac{a_0}{2} + \frac{1}{2} \\ a_{k+2} &= -\frac{a_k}{k+2} \end{aligned}$$

Plug in enough values of k to see a pattern and determine a_k.

$$\begin{aligned}
k = 1: \quad & a_3 = -\frac{a_1}{3} = -\frac{1}{3} \\
k = 2: \quad & a_4 = -\frac{a_2}{4} = \frac{a_0}{4 \cdot 2} - \frac{1}{4 \cdot 2} \\
k = 3: \quad & a_5 = -\frac{a_3}{5} = \frac{1}{5 \cdot 3} \\
k = 4: \quad & a_6 = -\frac{a_4}{6} = -\frac{a_0}{6 \cdot 4 \cdot 2} + \frac{1}{6 \cdot 4 \cdot 2} \\
k = 5: \quad & a_7 = -\frac{a_5}{7} = -\frac{1}{7 \cdot 5 \cdot 3} \\
& \vdots \\
& a_{2k} = (-1)^k \left[\frac{a_0}{(2k)!!} - \frac{1}{(2k)!!}\right] = (-1)^k \frac{a_0 - 1}{(2k)!!} = (-1)^k \frac{a_0 - 1}{2^k k!} = a_0 \frac{(-1)^k}{2^k k!} - \frac{(-1)^k}{2^k k!} \\
& a_{2k-1} = \frac{(-1)^{k+1}}{(2k-1)!!} = \frac{(-1)^{k+1}}{\frac{(2k)!}{2^k k!}} = (-1)^{k+1} \frac{2^k k!}{(2k)!}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=1}^{\infty} a_{2k-1} x^{2k-1} \\
&= a_0 + \sum_{k=1}^{\infty} \left[a_0 \frac{(-1)^k}{2^k k!} - \frac{(-1)^k}{2^k k!} \right] x^{2k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= a_0 + \sum_{k=1}^{\infty} a_0 \frac{(-1)^k}{2^k k!} x^{2k} - \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= \sum_{k=0}^{\infty} a_0 \frac{(-1)^k}{2^k k!} x^{2k} + 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= 1 + a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= 1 + a_0 \sum_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= 1 + a_0 e^{-x^2/2} - e^{-x^2/2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= 1 + (a_0 - 1) e^{-x^2/2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1} \\
&= 1 + A_0 e^{-x^2/2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k k!}{(2k)!} x^{2k-1}.
\end{aligned}$$

Note that by substituting $k = p + 1$, this last sum here can be written as

$$\sum_{p+1=1}^{\infty} (-1)^{p+2} \frac{2^{p+1}(p+1)!}{[2(p+1)]!} x^{2(p+1)-1},$$

or

$$\sum_{p=0}^{\infty} (-1)^p \frac{2^{p+1}(p+1)!}{[2(p+1)]!} x^{2p+1}.$$

Integrating Factor Method

$$y' + xy = 1 + x$$

Now solve the ODE by using an integrating factor I.

$$I = \exp\left(\int^x s\,ds\right) = e^{x^2/2}$$

Multiply both sides by I.

$$e^{x^2/2}y' + xe^{x^2/2}y = (1+x)e^{x^2/2}$$

The left side can be written as $d/dx(Iy)$ by the product rule. Expand the right side.

$$\frac{d}{dx}(e^{x^2/2}y) = e^{x^2/2} + xe^{x^2/2}$$

Integrate both sides with respect to x.

$$e^{x^2/2}y = \int^x e^{s^2/2}\,ds + \int^x se^{s^2/2}\,ds + C$$

Use the substitution $u = s^2/2$ (which means $du = s\,ds$) in the second integral.

$$\begin{aligned} e^{x^2/2}y &= \int^x e^{s^2/2}\,ds + \int^{x^2/2} e^u\,du + C \\ &= \int^x e^{s^2/2}\,ds + e^{x^2/2} + C \end{aligned}$$

Divide both sides by $e^{x^2/2}$ to solve for y.

$$\begin{aligned} y(x) &= e^{-x^2/2}\int^x e^{s^2/2}\,ds + 1 + Ce^{-x^2/2} \\ &= e^{-x^2/2}\int^x \sum_{p=0}^{\infty}\frac{(s^2/2)^p}{p!}\,ds + 1 + Ce^{-x^2/2} \\ &= e^{-x^2/2}\int^x \sum_{p=0}^{\infty}\frac{s^{2p}}{2^p p!}\,ds + 1 + Ce^{-x^2/2} \\ &= e^{-x^2/2}\sum_{p=0}^{\infty}\frac{1}{2^p p!}\int^x s^{2p}\,ds + 1 + Ce^{-x^2/2} \\ &= e^{-x^2/2}\sum_{p=0}^{\infty}\frac{1}{2^p p!}\frac{x^{2p+1}}{2p+1} + 1 + Ce^{-x^2/2} \\ &= \left[\sum_{p=0}^{\infty}\frac{(-x^2/2)^p}{p!}\right]\sum_{p=0}^{\infty}\frac{x^{2p+1}}{2^p p!(2p+1)} + 1 + Ce^{-x^2/2} \\ &= \left[\sum_{p=0}^{\infty}(-1)^p\frac{x^{2p}}{2^p p!}\right]\sum_{p=0}^{\infty}\frac{x^{2p+1}}{2^p p!(2p+1)} + 1 + Ce^{-x^2/2} \end{aligned}$$

Multiply the two series together.

$$\begin{aligned}
y(x) &= \sum_{p=0}^{\infty}\sum_{m=0}^{p}(-1)^m \frac{x^{2m}}{2^m m!}\frac{x^{2(p-m)+1}}{2^{p-m}(p-m)![2(p-m)+1]} + 1 + Ce^{-x^2/2} \\
&= \sum_{p=0}^{\infty}\sum_{m=0}^{p}(-1)^m \frac{x^{2p+1}}{2^p m!(p-m)!(2p-2m+1)} + 1 + Ce^{-x^2/2} \\
&= \sum_{p=0}^{\infty}\left[\frac{1}{2^p}\sum_{m=0}^{p}\frac{(-1)^m}{m!(p-m)!(2p-2m+1)}\right]x^{2p+1} + 1 + Ce^{-x^2/2}
\end{aligned}$$

All that remains is to show (using mathematical induction or the uniqueness of y, for example) that

$$\frac{1}{2^p}\sum_{m=0}^{p}\frac{(-1)^m}{m!(p-m)!(2p-2m+1)} = (-1)^p\frac{2^{p+1}(p+1)!}{[2(p+1)]!},$$

or

$$\sum_{m=0}^{p}\frac{(-1)^m}{m!(p-m)!(2p-2m+1)} = (-1)^p\frac{2^{2p+1}(p+1)!}{[2(p+1)]!}.$$

Then it will follow that

$$y(x) = \sum_{p=0}^{\infty}(-1)^p\frac{2^{p+1}(p+1)!}{[2(p+1)]!}x^{2p+1} + 1 + Ce^{-x^2/2}.$$