Problem 24

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \geq 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 23, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.

(b) Plot the graphs of $P_0(x), \ldots, P_5(x)$ for $-1 \leq x \leq 1$.

(c) Find the zeros of $P_0(x), \ldots, P_5(x)$.

Solution

The first few polynomial solutions of the Legendre equation were found in Problem 23 to be

$$\begin{aligned}
\alpha = 0: &\quad y(x) = 1 \\
\alpha = 1: &\quad y(x) = x \\
\alpha = 2: &\quad y(x) = 1 - 3x^2 \\
\alpha = 3: &\quad y(x) = x - \frac{5}{3}x^3 \\
\alpha = 4: &\quad y(x) = 1 - 10x^2 + \frac{35}{3}x^4 \\
\alpha = 5: &\quad y(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5.
\end{aligned}$$

Because the Legendre equation is homogeneous, any constant multiple of y is also a solution. Choose constants so that $y(1) = 1$ is satisfied for each value of α. The solutions for $\alpha = 0$ and $\alpha = 1$ already satisfy $y(1) = 1$, so

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x.
\end{aligned}$$

For $\alpha = 2$,

$$C_2 y(1) = C_2(1-3) = 1 \quad \rightarrow \quad C_2 = -\frac{1}{2},$$

which means

$$P_2(x) = -\frac{1}{2}(1 - 3x^2).$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

For $\alpha = 3$,

$$C_3 y(1) = C_3\left(1 - \frac{5}{3}\right) = 1 \quad \rightarrow \quad C_3 = -\frac{3}{2},$$

which means

$$P_3(x) = -\frac{3}{2}\left(x - \frac{5}{3}x^3\right).$$

For $\alpha = 4$,

$$C_4 y(1) = C_4\left(1 - 10 + \frac{35}{3}\right) = 1 \quad \rightarrow \quad C_4 = \frac{3}{8},$$

which means

$$P_4(x) = \frac{3}{8}\left(1 - 10x^2 + \frac{35}{3}x^4\right).$$

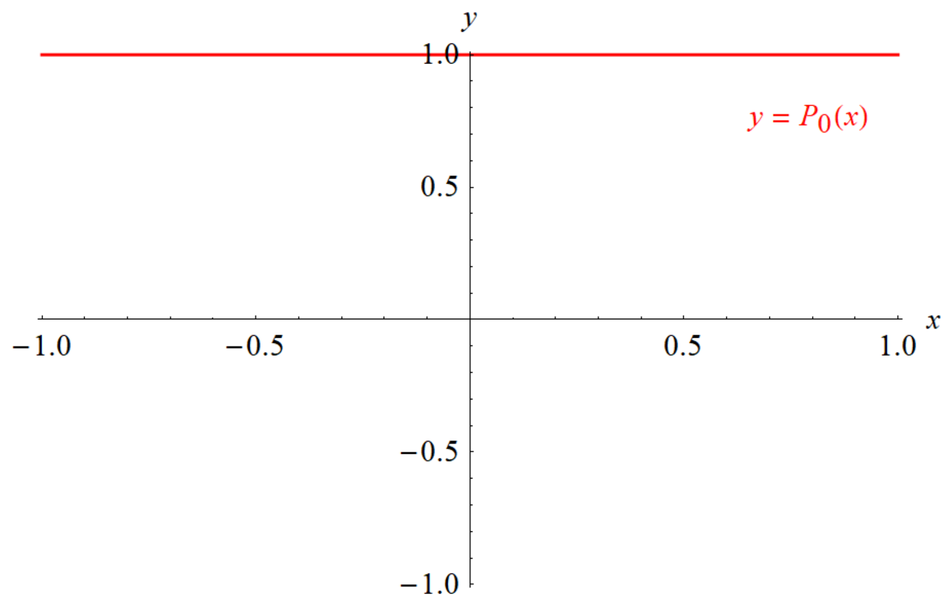
For $\alpha = 5$,

$$C_5 y(1) = C_5\left(1 - \frac{14}{3} + \frac{21}{5}\right) = 1 \quad \rightarrow \quad C_5 = \frac{15}{8},$$

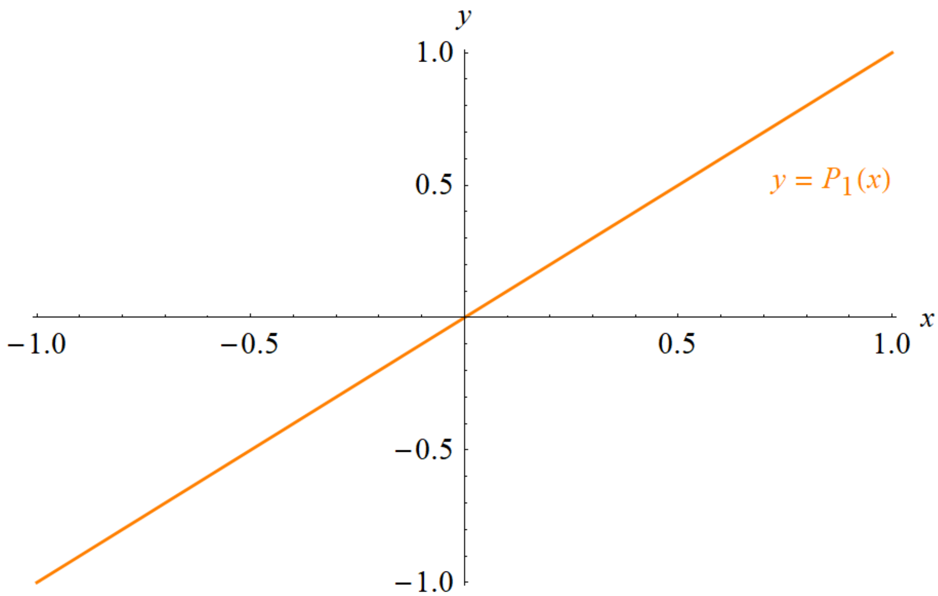
which means

$$P_5(x) = \frac{15}{8}\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right).$$

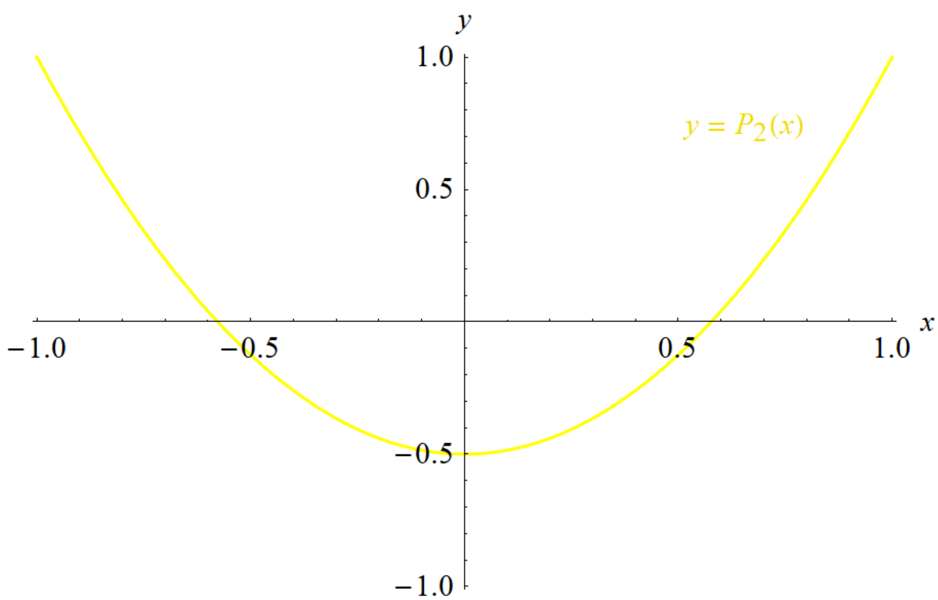
Below are the plots of these Legendre polynomials versus x for $-1 \leq x \leq 1$.



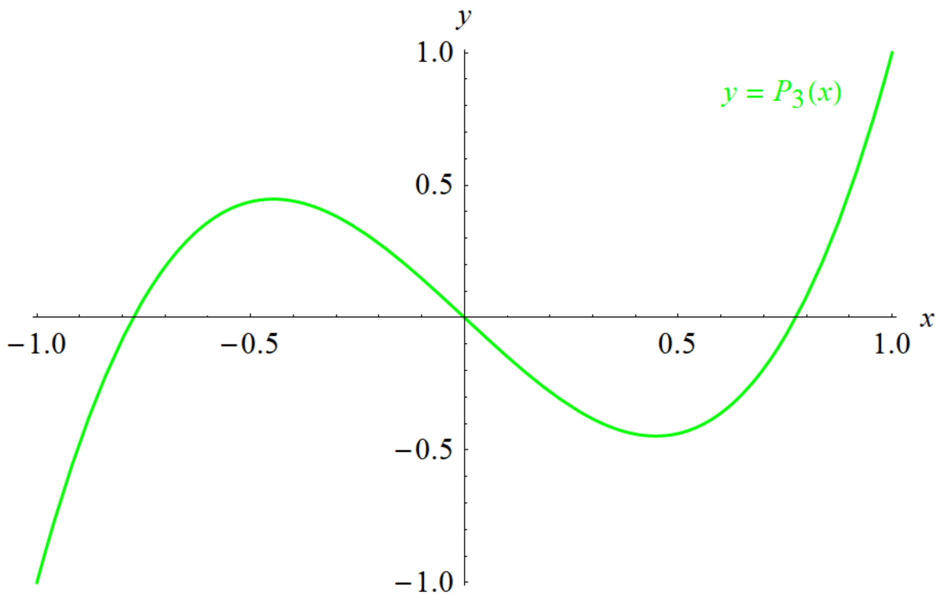
$P_0(x)$ has no zeros.



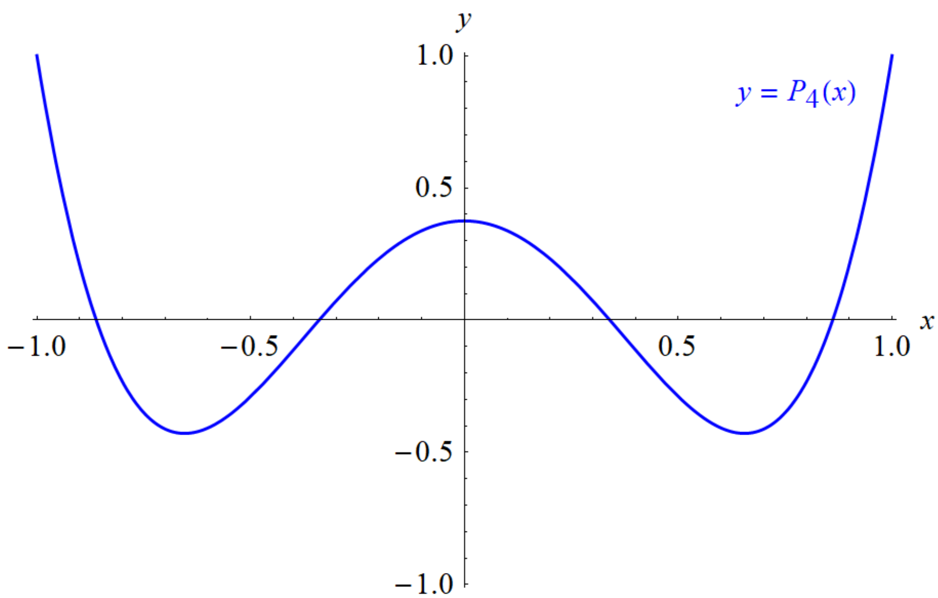
$P_1(x)$ has one zero at $x = 0$.



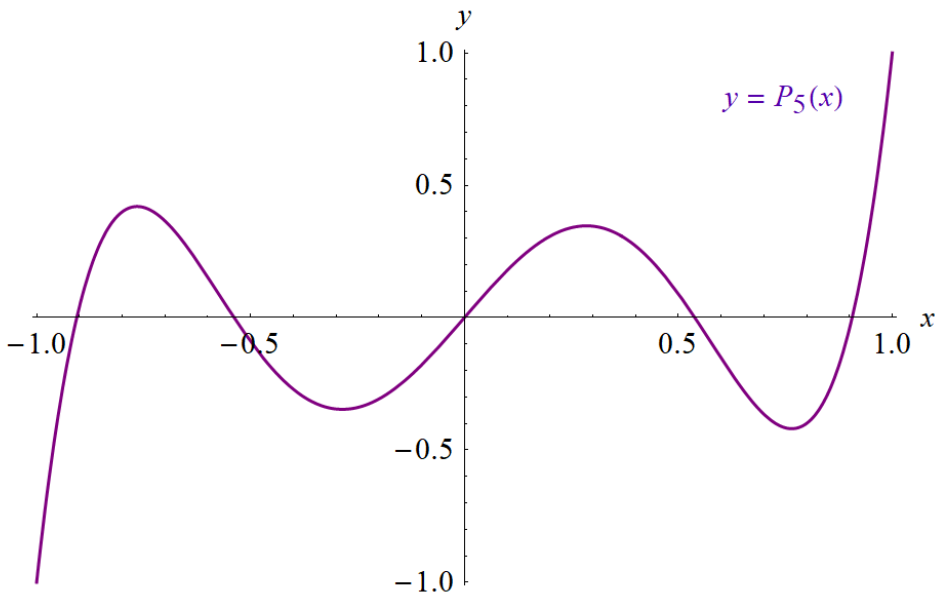
$P_2(x)$ has two zeros at $x = \pm 1/\sqrt{3} \approx \pm 0.577$.



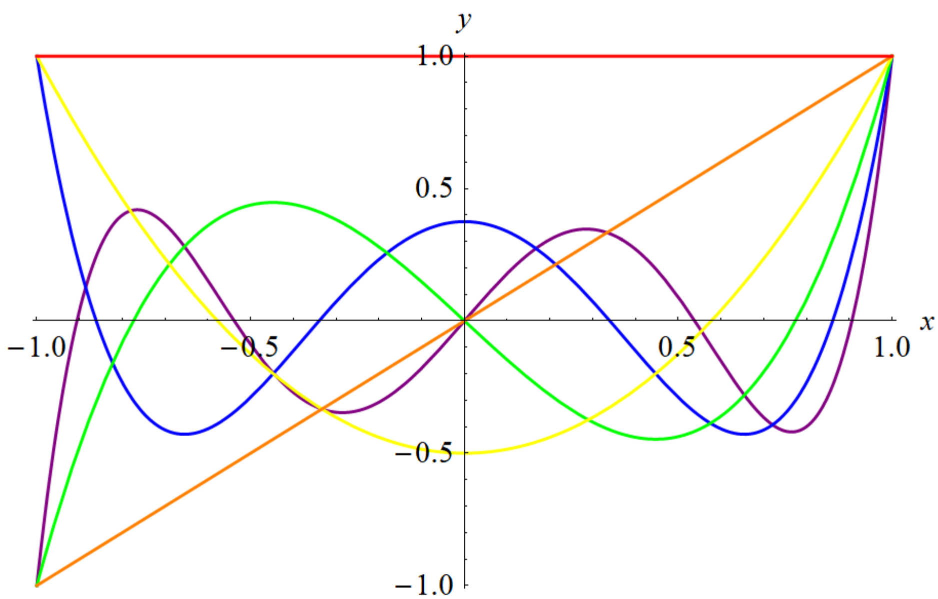
$P_3(x)$ has three zeros at $x = 0$ and $x = \pm\sqrt{3/5} \approx \pm 0.775$.



$P_4(x)$ has four zeros at $x = \pm\sqrt{(15 - 2\sqrt{30})/35} \approx \pm 0.340$ and $x = \pm\sqrt{(15 + 2\sqrt{30})/35} \approx \pm 0.861$.



$P_5(x)$ has five zeros at $x = 0$ and $x = \pm\frac{1}{3}\sqrt{(35 - 2\sqrt{70})/7} \approx \pm 0.538$ and $x = \pm\frac{1}{3}\sqrt{(35 + 2\sqrt{70})/7} \approx \pm 0.906$. All the previous graphs are superimposed on the one below.



Note that $P_0(x)$, $P_2(x)$, and $P_4(x)$ are even, whereas $P_1(x)$, $P_3(x)$, and $P_5(x)$ are odd.