Problem 26

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \qquad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

Solution

Rewrite cotangent as cosine over sine.

$$\frac{d^2F(\varphi)}{d\varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0$$

Now make the substitution $x = \cos\varphi$.

$$\frac{d^2F(\varphi)}{d\varphi^2} + \frac{x}{\sin\varphi}\frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0$$

Use the chain rule to write the derivatives in terms of this new variable.

$$\frac{dF}{d\varphi} = \frac{dF}{dx}\frac{dx}{d\varphi} = \frac{dF}{dx}(-\sin\varphi) \quad \to \quad \frac{1}{\sin\varphi}\frac{dF}{d\varphi} = -\frac{dF}{dx}$$

$$\begin{aligned}
\frac{d^2F}{d\varphi^2} = \frac{d}{d\varphi}\left(\frac{dF}{d\varphi}\right) = \frac{d}{d\varphi}\left[\frac{dF}{dx}(-\sin\varphi)\right] &= \frac{d}{d\varphi}\left(\frac{dF}{dx}\right)(-\sin\varphi) + \frac{dF}{dx}(-\cos\varphi) \\
&= \frac{dx}{d\varphi}\frac{d}{dx}\left(\frac{dF}{dx}\right)(-\sin\varphi) - x\frac{dF}{dx} \\
&= (-\sin\varphi)\frac{d^2F}{dx^2}(-\sin\varphi) - x\frac{dF}{dx} \\
&= \sin^2\varphi\frac{d^2F}{dx^2} - x\frac{dF}{dx} \\
&= (1-\cos^2\varphi)\frac{d^2F}{dx^2} - x\frac{dF}{dx} \\
&= (1-x^2)\frac{d^2F}{dx^2} - x\frac{dF}{dx}
\end{aligned}$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Therefore, the ODE in terms of x is

$$\left[(1-x^2)\frac{d^2F}{dx^2} - x\frac{dF}{dx}\right] + x\left(-\frac{dF}{dx}\right) + n(n+1)F = 0,$$

or

$$(1-x^2)\frac{d^2F}{dx^2} - 2x\frac{dF}{dx} + n(n+1)F = 0,$$

which is the Legendre equation with $\alpha = n$. Because n is a positive integer, the solution to this equation is the Legendre polynomial: $F(x) = P_n(x) = P_n(\cos\varphi) = F(\varphi)$.