Problem 28

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\alpha(\alpha+1)y.$$

Then it follows that

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \quad \text{and} \quad [(1-x^2)P_m'(x)]' = -m(m+1)P_m(x).$$

By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad \text{if} \quad n \neq m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the preceding integral is $2/(2n+1)$.

Solution

The Legendre equation is

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

Bring the third term to the right side.

$$(1-x^2)y'' - 2xy' = -\alpha(\alpha+1)y$$

The left side can be written as $d/dx[(1-x^2)y']$ by the product rule.

$$\frac{d}{dx}\left[(1-x^2)y'\right] = -\alpha(\alpha+1)y$$

Suppose that α is a positive integer n.

$$\frac{d}{dx}\left[(1-x^2)y'\right] = -n(n+1)y$$

Then the solution to the equation is the Legendre polynomial $y = P_n(x)$.

$$\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] = -n(n+1)P_n(x) \tag{1}$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Suppose that α is a different integer m.

$$\frac{d}{dx}\left[(1-x^2)y'\right] = -m(m+1)y$$

Then the solution to the equation is the Legendre polynomial $y = P_m(x)$.

$$\frac{d}{dx}\left[(1-x^2)P_m'(x)\right] = -m(m+1)P_m(x) \tag{2}$$

Multiply both sides of equation (1) by $P_m(x)$ and multiply both sides of equation (2) by $P_n(x)$.

$$P_m(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] = -n(n+1)P_n(x)P_m(x)$$
$$P_n(x)\frac{d}{dx}\left[(1-x^2)P_m'(x)\right] = -m(m+1)P_n(x)P_m(x)$$

Integrate both sides of each equation with respect to x from -1 to 1.

$$\int_{-1}^{1} P_m(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] dx = \int_{-1}^{1} -n(n+1)P_n(x)P_m(x)\, dx$$
$$\int_{-1}^{1} P_n(x)\frac{d}{dx}\left[(1-x^2)P_m'(x)\right] dx = \int_{-1}^{1} -m(m+1)P_n(x)P_m(x)\, dx$$

Use integration by parts for the integrals on the left and bring the constants in front of the integrals on the right.

$$P_m(x)\left[(1-x^2)P_n'(x)\right]\Big|_{-1}^{1} - \int_{-1}^{1} \frac{dP_m}{dx}(1-x^2)P_n'(x)\, dx = -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx$$
$$P_n(x)\left[(1-x^2)P_m'(x)\right]\Big|_{-1}^{1} - \int_{-1}^{1} \frac{dP_n}{dx}(1-x^2)P_m'(x)\, dx = -m(m+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx$$

$$0 - \int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\, dx = -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx$$
$$0 - \int_{-1}^{1} (1-x^2)P_n'(x)P_m'(x)\, dx = -m(m+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx$$

Now subtract both sides of the second equation from those of the first.

$$0 = -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx + m(m+1)\int_{-1}^{1} P_n(x)P_m(x)\, dx$$

Factor the integral.

$$0 = \left[-n(n+1) + m(m+1)\right]\int_{-1}^{1} P_n(x)P_m(x)\, dx$$

Therefore, provided that $n \neq m$,

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0.$$

The Legendre polynomials are said to be orthogonal.