Problem 4

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' + 3xy' + 5y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \to \quad y' = rx^{r-1} \quad \to \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2 r(r-1)x^{r-2} + 3xrx^{r-1} + 5x^r = 0$$

$$r(r-1)x^r + 3rx^r + 5x^r = 0$$

Divide both sides by x^r.

$$r(r-1) + 3r + 5 = 0$$

Solve for r.

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to the ODE are $y = x^{-1-2i}$ and $y = x^{-1+2i}$. According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$\begin{aligned}
y(x) &= C_1 x^{-1-2i} + C_2 x^{-1+2i} \\
&= C_1 x^{-1} x^{-2i} + C_2 x^{-1} x^{2i} \\
&= C_1 x^{-1} e^{\ln x^{-2i}} + C_2 x^{-1} e^{\ln x^{2i}} \\
&= C_1 x^{-1} e^{-2i \ln x} + C_2 x^{-1} e^{2i \ln x} \\
&= C_1 x^{-1}[\cos(-2\ln x) + i \sin(-2 \ln x)] + C_2 x^{-1}[\cos(2 \ln x) + i \sin(2 \ln x)] \\
&= C_1 x^{-1}[\cos(2\ln x) - i \sin(2 \ln x)] + C_2 x^{-1}[\cos(2 \ln x) + i \sin(2 \ln x)] \\
&= (C_1 + C_2) x^{-1} \cos(2 \ln x) + (-iC_1 + iC_2) x^{-1} \sin(2 \ln x) \\
&= C_3 x^{-1} \cos(2 \ln x) + C_4 x^{-1} \sin(2 \ln x)
\end{aligned}$$