Problem 6

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$(x-1)^2y'' + 8(x-1)y' + 12y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = (x-1)^r$.

$$y = (x-1)^r \quad \rightarrow \quad y' = r(x-1)^{r-1} \quad \rightarrow \quad y'' = r(r-1)(x-1)^{r-2}$$

Substitute these expressions into the ODE.

$$(x-1)^2 r(r-1)(x-1)^{r-2} + 8(x-1)r(x-1)^{r-1} + 12(x-1)^r = 0$$

$$r(r-1)(x-1)^r + 8r(x-1)^r + 12(x-1)^r = 0$$

Divide both sides by $(x-1)^r$.

$$r(r-1) + 8r + 12 = 0$$

Solve for r.

$$r^2 + 7r + 12 = 0$$

$$(r+4)(r+3) = 0$$

$$r = \{-4, -3\}$$

Two solutions to the ODE are $y = (x-1)^{-4}$ and $y = (x-1)^{-3}$. According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$y(x) = C_1(x-1)^{-4} + C_2(x-1)^{-3}.$$