Problem 7

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' + 6xy' - y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} + 6xrx^{r-1} - x^r = 0$$

$$r(r-1)x^r + 6rx^r - x^r = 0$$

Divide both sides by x^r.

$$r(r-1) + 6r - 1 = 0$$

Solve for r.

$$r^2 + 5r - 1 = 0$$

$$r = \frac{-5 \pm \sqrt{25 - 4(1)(-1)}}{2(1)} = \frac{-5 \pm \sqrt{29}}{2}$$

$$r = \left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$$

Two solutions to the ODE are $y = x^{(-5-\sqrt{29})/2}$ and $y = x^{(-5+\sqrt{29})/2}$. According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$y(x) = C_1x^{(-5-\sqrt{29})/2} + C_2x^{(-5+\sqrt{29})/2}.$$