Problem 8

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$2x^2y'' - 4xy' + 6y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$2x^2 r(r-1)x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

$$2r(r-1)x^r - 4rx^r + 6x^r = 0$$

Divide both sides by x^r.

$$2r(r-1) - 4r + 6 = 0$$

Solve for r.

$$2r^2 - 6r + 6 = 0$$

$$r^2 - 3r + 3 = 0$$

$$r = \frac{3 \pm \sqrt{9 - 4(1)(3)}}{2(1)} = \frac{3 \pm \sqrt{-3}}{2} = \frac{3 \pm i\sqrt{3}}{2} = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$$

$$r = \left\{\frac{3}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\right\}$$

Two solutions to the ODE are $y = x^{3/2 - i\sqrt{3}/2}$ and $y = x^{3/2 + i\sqrt{3}/2}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y(x) &= C_1 x^{3/2 - i\sqrt{3}/2} + C_2 x^{3/2 + i\sqrt{3}/2} \\
&= C_1 x^{3/2} x^{-i\sqrt{3}/2} + C_2 x^{3/2} x^{i\sqrt{3}/2} \\
&= C_1 x^{3/2} e^{\ln x^{-i\sqrt{3}/2}} + C_2 x^{3/2} e^{\ln x^{i\sqrt{3}/2}} \\
&= C_1 x^{3/2} e^{(-i\sqrt{3}/2)\ln x} + C_2 x^{3/2} e^{(i\sqrt{3}/2)\ln x} \\
&= C_1 x^{3/2}\left[\cos\left(-\frac{\sqrt{3}}{2}\ln x\right) + i\sin\left(-\frac{\sqrt{3}}{2}\ln x\right)\right] + C_2 x^{3/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + i\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] \\
&= C_1 x^{3/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) - i\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + C_2 x^{3/2}\left[\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + i\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] \\
&= (C_1 + C_2)x^{3/2}\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + (-iC_1 + iC_2)x^{3/2}\sin\left(\frac{\sqrt{3}}{2}\ln x\right) \\
&= C_3 x^{3/2}\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + C_4 x^{3/2}\sin\left(\frac{\sqrt{3}}{2}\ln x\right)
\end{aligned}$$