Problem 10

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$(x-2)^2y'' + 5(x-2)y' + 8y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = (x-2)^r$.

$$y = (x-2)^r \quad \rightarrow \quad y' = r(x-2)^{r-1} \quad \rightarrow \quad y'' = r(r-1)(x-2)^{r-2}$$

Substitute these expressions into the ODE.

$$(x-2)^2 r(r-1)(x-2)^{r-2} + 5(x-2)r(x-2)^{r-1} + 8(x-2)^r = 0$$

$$r(r-1)(x-2)^r + 5r(x-2)^r + 8(x-2)^r = 0$$

Divide both sides by $(x-2)^r$.

$$r(r-1) + 5r + 8 = 0$$

Solve for r.

$$r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$$

$$r = \{-2-2i, -2+2i\}$$

Two solutions to the ODE are $y = (x-2)^{-2-2i}$ and $y = (x-2)^{-2+2i}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y(x) &= C_1(x-2)^{-2-2i} + C_2(x-2)^{-2+2i} \\
&= C_1(x-2)^{-2}(x-2)^{-2i} + C_2(x-2)^{-2}(x-2)^{2i} \\
&= C_1(x-2)^{-2}e^{\ln(x-2)^{-2i}} + C_2(x-2)^{-2}e^{\ln(x-2)^{2i}} \\
&= C_1(x-2)^{-2}e^{-2i\ln(x-2)} + C_2(x-2)^{-2}e^{2i\ln(x-2)} \\
&= C_1(x-2)^{-2}\{\cos[-2\ln(x-2)] + i\sin[-2\ln(x-2)]\} \\
&\qquad + C_2(x-2)^{-2}\{\cos[2\ln(x-2)] + i\sin[2\ln(x-2)]\} \\
&= C_1(x-2)^{-2}\{\cos[2\ln(x-2)] - i\sin[2\ln(x-2)]\} \\
&\qquad + C_2(x-2)^{-2}\{\cos[2\ln(x-2)] + i\sin[2\ln(x-2)]\} \\
&= (C_1+C_2)(x-2)^{-2}\cos[2\ln(x-2)] + (-iC_1+iC_2)(x-2)^{-2}\sin[2\ln(x-2)] \\
&= C_3(x-2)^{-2}\cos[2\ln(x-2)] + C_4(x-2)^{-2}\sin[2\ln(x-2)]
\end{aligned}$$