Problem 12

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' - 4xy' + 4y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} - 4xrx^{r-1} + 4x^r = 0$$

$$r(r-1)x^r - 4rx^r + 4x^r = 0$$

Divide both sides by x^r.

$$r(r-1) - 4r + 4 = 0$$

Solve for r.

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$r = \{1, 4\}$$

Two solutions to the ODE are $y = x^1 = x$ and $y = x^4$. According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$y(x) = C_1x + C_2x^4.$$